

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 9 to 11

PAPER 13

QUASIRANDOMNESS

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

*Cover sheet
Treasury Tag
Script paper*

SPECIAL REQUIREMENTS

None

<p>You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.</p>

1 Let X and Y be sets of size m and n , respectively. Let $f : X \times Y \rightarrow [-1, 1]$.

Prove that the following two properties of f are equivalent, in a sense that you should explain:

(a)

$$\sum_{x, x' \in X} \sum_{y, y' \in Y} f(x, y)f(x, y')f(x', y)f(x', y') \leq c_1 m^2 n^2.$$

(b) For every subset $A \subset X$ and every subset $B \subset Y$,

$$\left| \sum_{x \in A} \sum_{y \in B} f(x, y) \right| \leq c_2 mn.$$

Prove that there exists a function f taking values ± 1 such that properties (a) and (b) hold, with constants c_i that converge to 0 as m and n tend to infinity.

2 State and prove some version of Szemerédi's regularity lemma. Use it to prove that for every $\delta > 0$ there exists N such that for every subset $A \subset \{1, 2, \dots, N\}^2$ of size at least δN^2 there exist x, y and $d \neq 0$ such that (x, y) , $(x, y + d)$ and $(x + d, y)$ belong to A .

3 (i) Let A, B, C be subsets of \mathbb{Z}_N with $|A| = \alpha N$, $|B| = \beta N$ and $|C| = \gamma N$. Suppose that the number of quadruples $(x, y, z, w) \in A^4$ such that $x + y = z + w \pmod{N}$ is at most $(\alpha^4 + c)N^3$. Prove that the number of triples $(a, b, c) \in A \times B \times C$ such that $a + c = 2b \pmod{N}$ differs from $\alpha\beta\gamma N^2$ by at most $f(c)$, where $f(c) \rightarrow 0$ as $c \rightarrow 0$.

(ii) Let $\delta > 0$ and let $A \subset \{1, 2, \dots, N\}$ be a set of size at least δN . Prove that $A + A + A$ contains an arithmetic progression of length at least N^r , where $r > 0$ depends on δ only.

END OF PAPER