UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 9.00 to 12.00

PAPER 48

QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Dirac equation is

$$(i\,\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) \;=\; 0$$

where the gamma matrices are given in the chiral representation by

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Here σ^i are the Pauli matrices and 1_2 is the unit 2×2 matrix.

a. Show that these matrices satisfy the Clifford algebra

$$\{\gamma^{\mu}, \gamma^{\nu}\} \;=\; 2\,\eta^{\mu\nu}\,1_4$$

where $\eta^{\mu\nu}$ is the Minkowski metric with signature (+1, -1, -1, -1) and 1_4 is the unit 4×4 matrix.

b. Let $\mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}$ be the generators of the Lorentz group. They satisfy the Lie algebra

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] \;=\; \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}.$$

Explain how to use the Clifford algebra to construct a representation of this Lie algebra, and show that the generators do indeed satisfy the commutation relations.

c. A Lorentz transformation is given by

$$\Lambda \;=\; \exp\left(\tfrac{1}{2}\,\Omega_{\rho\sigma}\,\mathcal{M}^{\rho\sigma}\right).$$

Describe the action on the spinor field $\psi(x)$. Explain why the spinor representation of the Lorentz group cannot be unitary. Explain why this means that $\psi^{\dagger}\psi$ is not a Lorentz scalar. How can one construct a Lorentz scalar from a spinor?

d. The charge conjugate of a spinor is defined to be

$$\psi^{(c)} \;=\; C\,\psi^{\star}$$

where C is a 4×4 matrix satisfying

$$C^{\dagger}C \;=\; 1, \qquad C^{\dagger}\,\gamma^{\mu}\,C \;=\; -\,(\gamma^{\mu})^{\star}.$$

Show that $\psi^{(c)}$ transforms in the same way as ψ under an infinitesimal Lorentz transformation. Show that if ψ satisfies the Dirac equation, then so does $\psi^{(c)}$.

2 A free real scalar field ϕ of mass μ may be expanded in the Heisenberg picture as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{i\,p\cdot x} \right)$$

where $E_{\vec{p}} = \sqrt{\vec{p}\cdot\vec{p} + \mu^2}$. The creation and annihilation operators obey the commutation relations

$$[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0 \quad \text{and} \quad [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\,.$$

a. Explain what is meant by the normal ordered product $:\phi(x)\,\phi(y):$ and the time ordered product $T(\phi(x)\,\phi(y))$.

The Feynman propagator is defined to be $\Delta_F(x-y) = \langle 0\,|\,T(\phi(x)\,\phi(y))\,|\,0\rangle$, where $|0\rangle$ is the vacuum state. Show that $\Delta_F(x-y)$ has the integral representation

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-\mu^2}\, e^{-ip\cdot(x-y)}$$

for a suitably chosen contour. Derive a relationship between the Feynman propagator and the normal ordered and time ordered products.

The Lagrangian density for pseudoscalar Yukawa theory is given by

$$\mathcal{L} = \tfrac{1}{2}\,(\partial\phi)^2 - \tfrac{1}{2}\,\mu^2\phi^2 + \bar{\psi}\,(i\,\gamma^\mu\,\partial_\mu - m)\psi - \lambda\,\phi\,\bar{\psi}\,\gamma^5\psi\,.$$

b. State the Feynman rules for the scalar propagator, the spinor propagator, and the interaction vertex.

c. Draw the lowest order Feynman diagrams for $\psi\psi \to \psi\psi$ scattering and $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering. Assign appropriate momentum and spin labels to the incoming and outgoing states and, using Feynman rules or otherwise, write down the amplitude for these two processes.

[*Useful Information:* The Feynman rules state that to each incoming fermion with momentum p and spin state r, you should associate the spinor $u^r(\vec{p})$. For outgoing fermions, associate $\bar{u}^r(\vec{p})$. To each incoming anti-fermion with momentum p and spin state r, associate a spinor $\bar{v}^r(\vec{p})$. For outgoing anti-fermions, associate $v^r(\vec{p})$.]

3 **a.** The Lagrangian density for Maxwell theory is

$$\mathcal{L} \;=\; -\frac{1}{4}\,F_{\mu\,\nu}\,F^{\mu\,\nu}$$

where $F_{\mu\,\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu$. Describe the gauge invariance of the theory. Derive the equation of motion for A_ν . Show that, with the choice of Lorentz gauge $\partial_\mu A^\mu = 0$, the equation of motion reduces to

$$(\partial_\mu\,\partial^\mu)\,A_\nu \;=\; 0\,.$$

b. Show that the equation of motion in Lorentz gauge follows from

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\,\nu}\,F^{\mu\,\nu} - \tfrac{1}{2}\,(\partial_\mu A^\mu)^2\,.$$

Compute the momenta $\pi^{\,\mu}$ conjugate to A_μ from this Lagrangian density.

c. The mode expansions for A_μ and $\pi^{\,\mu}$ in the Schrödinger picture are given by

$$A_\mu(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\,|\vec{p}|}}\,\sum_{\lambda=0}^{3}\epsilon_\mu^{\lambda}\,(\vec{p})\,\left[a_{\vec{p}}^{\lambda}\,e^{\,i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\lambda\,\dagger}e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\pi^{\,\mu}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\,\sqrt{\frac{|\vec{p}|}{2}}\,(+i)\,\sum_{\lambda=0}^{3}(\epsilon^{\,\mu})^{\lambda}\,(\vec{p})\,\left[a_{\vec{p}}^{\lambda}\,e^{\,i\vec{p}\cdot\vec{x}} - \,a_{\vec{p}}^{\lambda\,\dagger}\,e^{-i\vec{p}\cdot\vec{x}}\right]$$

where $\epsilon_\mu^{\lambda}(\vec{p})$ are four 4-vectors, satisfying $\epsilon^{\lambda}\cdot\epsilon^{\lambda'} = \eta^{\,\lambda\,\lambda'}$ and $(\epsilon_\mu)^{\lambda}(\epsilon_\nu)^{\lambda'}\eta_{\lambda\lambda'} = \eta_{\mu\nu}$. The creation and annihilation operators satisfy the commutation relations

$$[a_{\vec{p}}^{\lambda},\, a_{\vec{q}}^{\lambda'\,\dagger}] \;=\; -\,\eta^{\,\lambda\lambda'}(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q}) \qquad \text{and} \qquad [a_{\vec{p}}^{\lambda},\, a_{\vec{q}}^{\lambda'}] \;=\; [a_{\vec{p}}^{\lambda\dagger},\, a_{\vec{q}}^{\lambda'\,\dagger}] \;=\; 0\,.$$

Show that this implies canonical commutation relations for A_μ and π^{ν} .

d. The Lorentz invariant vacuum is defined by $a_{\vec{p}}^{\lambda}\,|\,0\rangle = 0$. What is wrong with the Hilbert space spanned by the one-particle states $a_{\vec{p}}^{\lambda\,\dagger}|\,0\rangle$? Explain how the Gupta-Bleuler condition implements the Lorentz gauge condition $\partial_\mu A^\mu = 0$, and sketch how this can be used to remove the difficulty with the states $a_{\vec{p}}^{\lambda\,\dagger}|\,0\rangle$. Identify the states with longitudinal polarization, and with transverse polarization.

4 Write an essay on symmetries in field theory. Your essay should: describe and prove Noether's theorem; provide illustrative examples of important symmetries in different field theories and explain their physical significance; describe the difference between a gauge symmetry and a global symmetry.

END OF PAPER