UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 9 to 12

PAPER 49

QUANTUM FIELD THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The free Klein-Gordon field $\phi(\mathbf{x}, t)$ obeys the equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0\,.$$

Using Noether's theorem, find the expression for the conserved 3-momentum $\mathbf{P}$.

In the quantized Klein-Gordon theory, the field $\phi(\mathbf{x})$ and the conjugate field $\pi(\mathbf{x})$ (in the Schrödinger representation) have the coupled expansions

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger\, e^{-i\mathbf{p}\cdot\mathbf{x}})$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger\, e^{-i\mathbf{p}\cdot\mathbf{x}})$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}\cdot\mathbf{p} + m^2}$ and $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ satisfy

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a_{\mathbf{p}}^\dagger\,,\ a_{\mathbf{p}'}^\dagger] = 0$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')\,.$$

Show that the 3-momentum operator $\mathbf{P}$ in the quantized theory can be expressed as

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\, \mathbf{p}\, a_{\mathbf{p}}^\dagger\, a_{\mathbf{p}}\,.$$

Calculate $[\mathbf{P}, a_{\mathbf{q}}^\dagger]$ and hence determine

$$e^{-i\mathbf{P}\cdot\mathbf{y}} a_{\mathbf{q}}^\dagger\, e^{i\mathbf{P}\cdot\mathbf{y}}$$

where $\mathbf{y}$ is a constant vector. What can you deduce about $e^{-i\mathbf{P}\cdot\mathbf{y}}|\mathbf{q}\rangle$, where $|\mathbf{q}\rangle$ denotes the one-particle state of 3-momentum $\mathbf{q}$?

Find $e^{-i\mathbf{P}\cdot\mathbf{y}}\phi(\mathbf{x})e^{i\mathbf{P}\cdot\mathbf{y}}$, and interpret your result.

2 In the chiral representation, the Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The matrix γ^5 is defined by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 .$$

Calculate γ^5 and show that it anticommutes with γ^0 and γ^i.

Consider the massless Dirac equation

$$i\gamma^\mu\partial_\mu\psi = 0$$

for a left-handed spinor field, satisfying $\gamma^5\psi = -\psi$. A plane wave solution is of the form

$$\psi(x) = \lambda(p)e^{-ip\cdot x} .$$

What condition does the 4-momentum p^μ have to satisfy for such a solution to exist? Find and solve the equation for $\lambda(p)$, assuming the condition is satisfied.

Find the effect on this solution of a spatial rotation around the axis parallel to the 3-momentum $\mathbf{p}$. What can you deduce about the spin states of particles in the quantized theory of a left-handed Dirac field?

3 The interaction picture field $\phi(x)$ of a quantized scalar field theory satisfies the relation

$$\mathrm{T}\phi(x)\phi(y) = \ :\phi(x)\phi(y): \ + \ D_F(x-y)\,.$$

Explain the meaning of the various expressions occurring here, and establish this relation.

Explain in outline how you would derive the Feynman rules for correlation functions

$$\langle 0|\mathrm{T}(\,\phi(x_1)\phi(x_2)\ldots\phi(x_n)S\,)|0\rangle\,,$$

where S is the S-matrix, in the theory whose Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\mu}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4\,.$$

For what range of parameter values do you expect this theory to have a stable vacuum?

$$\left[\ \phi(x)\ \textit{ has the expansion }\ \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x})\,.\ \right]$$

4 Write notes on:

(a) Gauge invariance of a classical electromagnetic field coupled to matter fields.

(b) The effect of position-independent gauge transformations, $\psi(x) \to e^{ie\chi}\psi(x)$, on the Feynman rules for QED scattering amplitudes, and the relationship with charge conservation.

(c) Gauge invariance and the photon propagator.

(d) The Landau gauge $\partial_\mu A^\mu = 0$, and the photon propagator in Landau gauge

$$-\frac{i}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right).$$

END OF PAPER