UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 9 to 12

PAPER 62

QUANTUM FIELD THEORY

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A hermitian scalar, $\phi(x)$, has a Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}(\partial\phi(x))^2 - \frac{1}{2}m^2\phi^2(x) \ .$$

Explain briefly the procedure for quantising the field theory and show that in the Heisenberg Picture,

$$[\phi(\mathbf{x},t), \dot{\phi}(\mathbf{x}',t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}') \ .$$

Construct the Hamiltonian for the field theory and show that the Heisenberg fields obey

$$(\partial^2 + m^2)\phi(x) = 0 \ ,$$

and hence that $\phi(x)$ can be expressed in terms of mode operators, $a(p)$ and $a^\dagger(p)$, in the form

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left(a(p)e^{-ip.x} + a^\dagger(p)e^{ip.x}\right) \ ,$$

where $p^2 = m^2$ and $E_p = \sqrt{\mathbf{p}^2 + m^2}$. Write down and justify the commutation relations satisfied by the mode operators.

Express the Hamilitonian in terms of the mode operators and show that

$$[H, a(p)] = -E_p \ a(p) \quad , \qquad \text{and} \qquad [H, a^\dagger(p)] = E_p \ a^\dagger(p) \ .$$

Explain the particle interpretation of the theory.

The Feynman propagator, $\Delta_F(x-y)$, for the field $\phi(x)$ is defined by the equation

$$i\Delta_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle \ ,$$

where the symbol T indicates the time ordered product. Show that

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m^2 + i\epsilon}e^{-ip.(x-y)} \ ,$$

and hence that

$$(\partial^2 + m^2)\Delta_F(x-y) = -\delta^{(4)}(x-y) \ .$$

2 A non-hermitian scalar field, $\phi(x)$, has a Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, where

$$\mathcal{L}_0(x) = \partial\phi^\dagger(x).\partial\phi(x) - m^2\phi^\dagger(x)\phi(x) \ ,$$

and

$$\mathcal{L}_I(x) = -\frac{\lambda}{4}\big(\phi^\dagger(x)\phi(x)\big)^2 \ .$$

Explain how this split of the Lagrangian density can be used to define the Interaction Picture in which the field $\phi(x)$ has the form

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\left(a(p)e^{-ip.x} + b^\dagger(p)e^{ip.x}\right) \ ,$$

where $a(p)$, $a^\dagger(p)$ and $b(p)$, $b^\dagger(p)$ have the usual commutation relations which you should state.

Show that in the interaction picture the operator that describes the evolution of the states of the system in time from the far past to the far future is S, where

$$S = T\exp\left\{i\int d^4x\,\mathcal{L}_I(x)\right\} \ .$$

Let $|0\rangle$ be the vacuum state and let

$$|p_1q_1\rangle = a^\dagger(p_1)b^\dagger(q_1)|0\rangle \qquad \text{and} \qquad |p_2q_2\rangle = a^\dagger(p_2)b^\dagger(q_2)|0\rangle \ .$$

Show that, for I the unit operator,

$$\langle p_2q_2|(S-I)|p_1q_1\rangle = i(2\pi)^4\delta^{(4)}(p_1+q_1-p_2-q_2)T \ ,$$

where, to $O(\lambda)$,

$$T = -\lambda \ .$$

Calculate $\langle p_2q_2|I|p_1q_1\rangle$.

3 The Dirac equation for a particle of mass m is

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \ ,$$

where $\psi(x)$ is the spinor wave function for the particle and the matrices, $\{\gamma^\mu\}$ are given by

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad , \qquad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \ .$$

Here $\mathbf{1}$ is the 2×2 unit matrix and $\{\sigma_j\}$ are the Pauli matrices which satisfy $\sigma_j^2 = \mathbf{1}$ and $\sigma_j \sigma_k = -\sigma_k \sigma_j = i\sigma_l$ where (j,k,l) is a cyclic permutation of $(1,2,3)$. Verify that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ ,$$

where $g^{\mu\nu}$ is the standard Lorentz metric. Use this result to show that $\psi(x)$ satisfies the Klein-Gordan equation for a relativistic particle of mass m.

Under a parity transformation a spacetime point $x \to x_{_P} = (x^0, -\mathbf{x})$, a 4-momentum $p \to p_{_P} = (E_p, -\mathbf{p})$ and a Dirac spinor wavefunction, $\psi(x)$ transforms according to

$$\psi(x) \to \psi_{_P}(x) = \gamma^0 \psi(x_{_P}) \ .$$

Verify that the Dirac equation is invariant under parity transformations.

Show that a solution of the Dirac equation corresponding to a particle with 4-momentum p has the form

$$\psi_{p,s}(x) = u_s(p) e^{-ip.x}$$

where

$$u_s(p) = \begin{pmatrix} \chi_s \\ \boldsymbol{\sigma}.\mathbf{p}\chi_s/(E_p + m) \end{pmatrix} \ ,$$

and χ_s is a two component spinor. Explain the significance of the label s . Verify that under the parity transformation

$$\psi_{p,s}(x) \to \psi_{p_{_P},s}(x) \ .$$

What does this imply about the spin state of the relativistic particle after a parity transformation?

Deduce the transformation of the conjugate Dirac spinor $\bar{\psi}(x)$ under parity transformations and obtain the transformations of the bilinears $\bar{\psi}(x)\psi(x)$ and $\bar{\psi}(x)\gamma^5\psi(x)$ under parity, where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Use the properties of the Dirac matrices to evaluate the quantities $\mathrm{Tr}(\gamma.p)$, $\mathrm{Tr}(\gamma.p\,\gamma.q)$, $\mathrm{Tr}(\gamma.p\,\gamma.q\,\gamma.k)$ and $\mathrm{Tr}(\gamma.p\,\gamma^\mu\,\gamma.q\,\gamma^\nu)$, where p, q and k are 4-vectors.

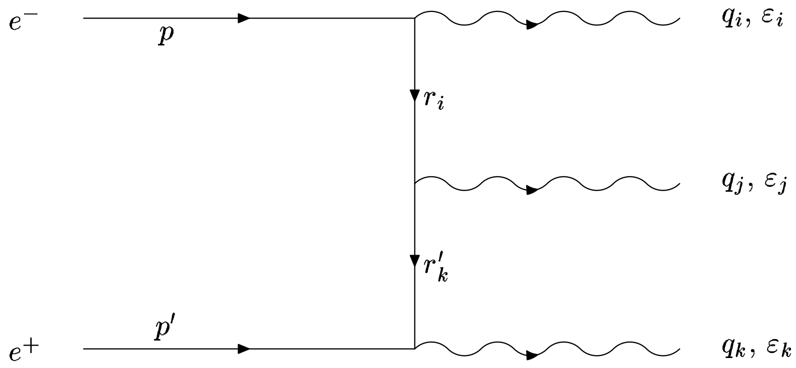
4 The amplitude, correct to $O(e^3)$, for the process

$$e^-(p) + e^+(p') \to \gamma(q_1) + \gamma(q_2) + \gamma(q_3) \ ,$$

is A where

$$A = \sum_{\text{perms}} T^{\mu\nu\sigma}_{ijk}\, \epsilon_{i\mu}\epsilon_{j\nu}\epsilon_{k\sigma} \ ,$$

ϵ_1, ϵ_2, ϵ_3 are the polarisation vectors of the final state photons, $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$ and the summation is over the six permutations. The amplitude $T^{\mu\nu\sigma}_{ijk}\epsilon_{i\mu}\epsilon_{j\nu}\epsilon_{k\sigma}$ is associated with the Feynman diagram



where $r_i = p - q_i$ and $r'_k = q_k - p' = r_i - q_j$ and

$$T^{\mu\nu\sigma}_{ijk} = (ie)^3 \bar{v}(p')\gamma^\sigma \frac{\gamma.r'_j + m}{r'^2_j - m^2}\gamma^\nu \frac{\gamma.r_i + m}{r_i^2 - m^2}\gamma^\mu u(p) \quad .$$

Show that

$$T^{\mu\nu\sigma}_{ijk}\, q_{i\mu} = -(ie)^3 \bar{v}(p')\gamma^\sigma \frac{\gamma.r'_j + m}{r'^2_j + m^2}\gamma^\nu u(p)$$

and that

$$T^{\mu\nu\sigma}_{ijk}\, q_{j\nu} = (ie)^3 \left(\bar{v}(p')\gamma^\sigma \frac{\gamma.r'_k}{r'^2_k - m^2}\gamma^\mu u(p) - \bar{v}(p')\gamma^\sigma \frac{\gamma.r_i}{r_i^2 - m^2}\gamma^\mu u(p) \right) \ .$$

Evaluate also $T^{\mu\nu\sigma}_{ijk}\, q_{k\sigma}$.

Draw the six diagrams that contribute to A and use these results to show that when the polarisation vector of a photon is replaced by its 4-momentum the amplitude A vanishes. What is the physical significance of this result?