UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 1.30 to 3.30

PAPER 52

POPULATION DYNAMICS

*Attempt at most **THREE** questions. The questions carry equal weight.*

Additional credit is given for substantially complete answers.

Wherever possible state any biological assumptions and give a biological interpretation to your results.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write down the Lotka-Volterra equations for a predator-prey model, find the Jacobian at the non-trivial fixed point and describe the dynamics close to this point.

A species of predator and prey live in temperate (seasonal) latitudes. For most of the year the only events that occur are predation and the natural death of predators and prey. During a very short time scale predator and prey reproduction occurs.

(a) Based on the Lotka-Volterra equations, write down a model to describe the dynamics of these populations. (You may ignore age-structure, and assume that the number of offspring a predator produces is proportional to the number of prey available at the end of the year.)

(b) Compare this model to the Nicholson-Bailey equations, find whether the predator-prey system has stable annual oscillations, and sketch the dynamics.

2 Two species of pig live in the forests of Borneo. Both species reproduce throughout the year, obey logistic dynamics (with different parameters) and have no natural interaction. Local hunters set traps for these pigs, which can be modelled as harvesting at some constant level, H. If one species of pig is easier to trap than the other, describe the range of possible behaviour for all levels of hunting, and plot equilibrium levels against H.

3 Compare and contrast macro-parasite and micro-parasite infections, paying particular attention to the mathematical tools necessary and the heterogeneities encountered. (Simple models should be used to illustrate the main points.)

4 A sexually transmitted disease (STD) can be described by an SIS (susceptible-infectious-susceptible) model. (births and deaths may be ignored throughout)

(a) Write down the appropriate model for the proportion of individuals who are infectious. Give the definition of R_0 in words. Find the relationship between R_0 and the equilibrium level of susceptibles for this model.

In reality the population can be divided into two groups. A very small proportion of individuals (p) have many sexual contacts, the remainder of the population have very few. These are classified as high-risk and low-risk groups.

(b) Write down new equations for the two populations, and explain how R_0 should be calculated. (You may find it easier to model the proportion of infectious individuals in each group.) Your model should reflect the case that individuals in either group have the majority of their contacts with other individuals in the same group, and that the number contacts between low and high risk groups are of order p.

(c) Expanding to first order in p, find R_0 and the equilibrium level of susceptibles, and comment on their relationship.

5 There are two alleles **A** and **a** which can exist at a locus. The heterozygote genotype has a lower fitness than the two homogeneous genotypes.

(a) Develop an equation for the proportion of each allele in the population, and discuss the dynamics.

This species is found to occupy a set of N distinct habitats (a metapopulation), with global coupling σ between the habitats.

(b) Given that initially $N - m$ habitats contain pure **aa** and m habitats contain pure **AA**, find the conditions which prevent fixation. (You may assume the two homogeneous genotypes have equal fitness. Hint: consider the situation when the rate of increase of **A** is maximal in m sub-populations.)

6 The payoffs in the prisoner's dilemma are,

For iterated prisoners dilemma, strategies $S = (C_C, C_D)$ define the probability of co-operating given that the other play cooperated (C_C) or defected (C_D) last time.

(a) Consider the long-term dynamics of a two-player game (you may assume that all strategies start as if the other player cooperated last time)

(b) Show that Tit-for-Tat ($C_C = 1$, $C_D = 0$) is an ESS, comment on the results and show that Tat-for-Tat can eventually invade all strategies.