MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2008 1.30 to 4.30

PAPER 73

PHYSICAL COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Explain the difference between kinematic and cosmological redshifts.

(ii) Use the relativistic formula

$$\nu_0 = \nu_{\rm e}\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}},$$

where $\nu_{\rm e}$ and ν_0 are the photon emission and reception frequencies respectively, to derive the kinematic redshift of a light source moving at velocity v relative to the observer, in the limit $v \ll c$.

(iii) Use the Robertson-Walker metric

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

to show that the quantity $(1 + z_{\rm cosm}) = (\nu_{\rm e}/\nu_0)$, where $z_{\rm cosm}$ is the cosmological redshift of a light source, is inversely proportional to the scale factor of the universe $a(t)$.

(iv) Consider two galaxies, A and B. As viewed from Earth, galaxy A is at redshift $z_{\rm A} = 1$ and galaxy B is at $z_{\rm B} = 9$. What is the redshift of galaxy B as measured by a hypothetical observer on galaxy A?

(v) When observing a distant galaxy, we measure a combination of kinematic and cosmological redshifts, as galaxies respond to the local gravitational field which adds a 'peculiar' velocity (of either positive or negative sign) on the uniform Hubble expansion. (a) Show that the combined redshift due to these effects [kinematic and cosmological] is

$$1 + z_{\rm tot} = (1 + z_{\rm cosm})(1 + z_{\rm kin})$$

(b) Typical peculiar velocities of galaxies are $v_{\rm p} = \pm 600\,{\rm km\ s^{-1}}$. For a constant Hubble parameter $H_0 = 100\,{\rm km\ s^{-1}\ Mpc^{-1}}$, what is the minimum distance, $r_{\rm min}$, at which a galaxy must be for its redshift to give an estimate of its true distance accurate to better than 5%?

(vi) The spectrum of blackbody radiation is described by the Planck function

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

where ν is the frequency, T is the temperature and h, k and c are constants.

(a) Knowing that $B_\nu(T)$ has a maximum at $\nu_{\rm max}$, deduce Wien's law: $\nu_{\rm max}/T = C$, where C is a constant. (b) Given that the Sun's peculiar velocity relative to the Hubble flow is $v_{\rm p} \simeq 300\,{\rm km\ s^{-1}}$, estimate the temperature of the microwave background radiation measured in the direction of the Sun's motion.

(vii) Explain what is meant by the K-correction in the calculation of the absolute magnitude M of an astronomical source at luminosity distance $d_{\rm L}$. Consider two cases where we measure the apparent magnitudes in the R-band, $m_{\rm R}$, of (a) an elliptical galaxy and (b) a young star-forming galaxy, both at redshift $z = 3$. If we neglect their K-corrections, would we underestimate or overestimate their true absolute magnitudes $M_{\rm R}$? The R-band is centred at a wavelength of 660 nm. Give reasons for your answers.

2 In a spatially flat Universe where radiation makes a negligible contribution to the energy density, the Friedmann equation can be written as:

$$\dot{a}^2 = H_0^2 \Omega_{\rm m,0} a^{-1} + H_0^2 \Omega_{\Lambda,0} a^2$$

where the subscript 0 denotes the present time, H is the Hubble parameter, a is the scale factor and $a_0 = 1$, and $\Omega_{\rm m}$ and Ω_Λ denote respectively the contributions of matter and the cosmological constant to the critical density.

(i) Derive an expression for the ratio of the Hubble parameter at redshift z to its value today.

(ii) Show that in this cosmology the age of the Universe at redshift z can be written as:

$$t(z) = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}} \ln\left[\frac{1+\cos\theta}{\sin\theta}\right]$$

where

$$\tan\theta = \left(\frac{\Omega_{\rm m,0}}{\Omega_{\Lambda,0}}\right)^{1/2} (1+z)^{3/2}$$

You may use the identity:

$$\int \frac{dx}{\sin x} = \ln\left[\frac{\sin x}{1+\cos x}\right]$$

(iii) Hence show that the present age of the Universe is given by

$$t_0 = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}} \ln\left[\frac{1+\Omega_{\Lambda,0}^{1/2}}{(1-\Omega_{\Lambda,0})^{1/2}}\right]$$

(iv) Recently a team of astronomers announced the discovery of a galaxy at redshift $z = 8$. This claim generated some controversy partly because, if true, it would imply that galaxies formed only a short time after the big bang. Assuming an Einstein-de Sitter universe ($\Omega_{\rm m,0} = 1, \Omega_{\Lambda,0} = \Omega_{\rm k,0} = 0$), calculate (a) the fraction of the current age of the universe at which this galaxy was observed; (b) the time interval available from the big bang to $z = 8$ to form this galaxy. (c) Would the interval available be longer or shorter in a $\Omega_{\rm m,0} = 0.3, \Omega_{\Lambda,0} = 0.7, \Omega_{\rm k,0} = 0$ cosmology? (Use $1/H_0 = 1 \times 10^{10} h^{-1}$ years for parts (a), (b) and (c) of this question.)

3 (i) Define the distribution of the neutral hydrogen column densities, $N(\text{H I})$, of Lyman alpha absorbers. Show its functional form with the aid of a sketch, indicating the ranges of $N(\text{H I})$ which are referred to as 'Lyman alpha forest', 'Lyman limit systems' and 'damped Lyman alpha systems'. Briefly explain the meaning of this terminology.

(ii) From the Friedmann equation:

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3c^2}\rho,$$

where a is the scale factor, k is the curvature, Λ is the cosmological constant, ρ is the density, the other symbols have their usual meaning and a dot denotes differentiation with respect to the proper time, show that for a uniform comoving population of absorbers, each with constant cross-section, the probability that the line of sight to a distant quasar intersects such an absorber per unit redshift at redshift z is proportional to:

$$\frac{(1+z)^2}{[\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{k},0}(1+z)^2 + \Omega_{\Lambda,0}]^{1/2}}$$

where $\Omega_{\mathrm{m},0}$ and $\Omega_{\mathrm{k},0}$ and $\Omega_{\Lambda,0}$ denote respectively the contributions of matter, curvature and the cosmological constant to the critical density at the present epoch.

(iii) In an Einstein-de Sitter universe, with $\Omega_{\mathrm{m},0} = 1$, $\Omega_{\mathrm{k},0} = \Omega_{\Lambda,0} = 0$, how would you expect, on the basis of your answer to (ii), the number of Lyman alpha absorbers—per unit redshift interval and with neutral hydrogen column density above a given threshold value—to change between $z = 1.5$ and $z = 4$? Briefly discuss how this compares with what is observed.

(iv) Explain with the aid of physical arguments how you would measure the temperature of the intergalactic medium at redshift $z = 3$ from the Lyman alpha lines seen in quasar spectra.

4 (i) Show that the primordial abundance of helium by mass is

$$Y_{\rm p} = 2\left(1 + \frac{n_{\rm p}}{n_{\rm n}}\right)^{-1}$$

where $n_{\rm p}/n_{\rm n}$ is the ratio (by number) of protons to neutrons, if all the baryons are in H and He.

(ii) Imagine another universe, described by the same cosmological parameters as our own, but with the one difference that a force of unknown origin decelerated the universal expansion between times $t_1 = 1\,{\rm s}$ and $t_2 = 300\,{\rm s}$. Would you expect the mass fraction of hydrogen in this alternate universe to be larger than, smaller than, or the same as, that in our universe? Justify your answer in a few sentences.

(iii) (a) Describe how the *primordial* abundances of ^{4}He, D, and ^{7}Li are deduced from observations. (b) Discuss, with the aid of a sketch, whether the current best estimates of these abundances are in agreement with the predictions of Big Bang Nucleosynthesis and the values of $\Omega_{\rm b,0}$ deduced from other experiments, where $\Omega_{\rm b,0}$ is the present-day contribution of baryons to the critical density. (c) From the above discussion draw your conclusions regarding the validity of the Big Bang cosmological framework.

END OF PAPER