UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 5 June 2003 9 to 12

PAPER 66

PHYSICAL COSMOLOGY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight, parts of questions may have different weights.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Discuss the observational evidence for the isotropy and homogeneity of the Universe.

(b) Show that the redshift z of a distant galaxy is related to the scale factor $a(t)$ at that epoch by $1 + z = 1/a$, assuming that at the present epoch $a(t_0) = 1$.

(c) Explain what is meant by proper and comoving coordinates.

(d) In a matter dominated Universe the scale factor $a(t)$ obeys

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3} G\rho + \frac{\Lambda}{3}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}\,.$$

Explain the meaning of the different terms in these equations. For $\Lambda = 0$ derive them from Newtonian theory. What Newtonian potential corresponds to the Λ-term?

(e) Consider a 'bouncing Universe', in which $a(t)$ was large at early times, then reached a minimum at a_*, and later grew with time. Define the conditions for $\dot{a}$ and $\ddot{a}$ at the bouncing redshift z_* and show that:

$$\Omega_{\mathrm{m}} \leqslant \frac{2}{z_*^2(z_* + 3)}\,.$$

Assume a quasar is observed at $z = 6$. What is the upper limit on Ω_{m} in the bouncing model?

Discuss briefly the observational evidence that $\Omega_{\mathrm{m}} \gtrsim 0.2$, and therefore explain why the bouncing model is ruled out.

2 (a) Consider a flat universe with just one component which has an equation of state $P_\phi = w\rho_\phi$ where w is a constant. Write down the Hubble parameter $H(t)$, the density $\rho(t)$ and the deceleration parameter $q(t)$ for such a universe.

What is the condition on w for an accelerated expansion?

(b) Consider a scalar field ϕ and a potential $V(\phi)$. Assume the pressure and the density are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,.$$

For the equation of state of Part (a), derive $\phi(a)$, where $a(t)$ is the scale factor, and show that

$$V(\phi) = \frac{1}{2}(1-w)\rho_\phi^0 \exp\left[-\sqrt{24\pi G(1+w)}(\phi - \phi_0)\right]$$

where ρ_ϕ^0 and ϕ_0 are constants.

Consider a scalar field model with a potential of the form

$$V(\phi) = V_0 e^{-\lambda\phi\sqrt{8\pi G}}$$

where λ is a positive constant. What are the constraints on λ for an accelerating universe which also satisfies the weak energy condition $\rho_\phi + P_\phi > 0$?

(c) If supernovae Type Ia are assumed to be standard candles, write down the flux ratios of different redshifts in terms of the Hubble parameter $H(z)$, and explain how this can be used to estimate the equation of state parameter w. Discuss the uncertainties in this approach.

3 (a) During the matter dominate era the density contrast $D(t)$ obeys in linear theory the differential equation

$$\ddot{D} + 2\frac{\dot{a}}{a}\dot{D} - \frac{3}{2}H_0^2\Omega_{\rm m}\frac{D}{a^3} = 0$$

where $a(t)$ is the scale factor, H_0 is the Hubble constant and $\Omega_{\rm m}$ is the present-epoch mass density parameter. Show that the Hubble parameter $H(t)$ is a solution to this equation in the matter-dominated era.

Obtain the two solutions to this equation in an Einstein-de Sitter universe.

Is $H(t)$ the growing or the decaying solution?

(b) Derive the relation between the two-point correlation function $\xi(r)$ and the power spectrum of the mass fluctuations $P(k)$.

Express your result as a one-dimensional integral over k.

Assume that $P(k) \propto k^n$, where n is a constant. By dimensional arguments derive the *rms* density and potential fluctuations in a sphere of radius R and the *rms* bulk flow of that space.

(c) By decomposing the fluctuations in the potential into Fourier components and using Rayleigh's expansion of a plane wave in spherical waves

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi\sum_{l,m}(i)^l j_l(kr)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{k}})$$

show that in linear theory the line-of-sight peculiar velocity is

$$U(\mathbf{r}) = \mathbf{v}(\mathbf{r})\cdot\hat{\mathbf{r}} = \frac{\Omega_{\rm m}^{0.6}}{2\pi^2}\sum_{l,m}(i^l)^*\int d^3\mathbf{k}\,\frac{\delta_{\mathbf{k}}}{k}\,\frac{dj_l(kr)}{d(kr)}Y_{lm}(\hat{\mathbf{k}})Y^*_{lm}(\hat{\mathbf{r}})\,.$$

4 (a) Discuss briefly the evidence for dark matter in clusters of galaxies based on (i) the velocity dispersion of cluster galaxies, (ii) the temperature of hot gas and (iii) gravitational lensing.

(b) Describe qualitatively and sketch a diagram of the evolution with time of the radius of a spherical proto-cluster in an Einstein-de Sitter universe ($\Omega_{\rm m} = 1, \Lambda = 0$) from an early time in its history to virialization.

Show that the radius of the virialized cluster is half the maximum radius.

What is the density of virialization relative to density at turn around?

Show that the density at collapse $\rho(z_{\rm coll})$ is proportional to $\rho_0(1 + z_{\rm coll})^3$, where $z_{\rm coll}$ is the redshift of collapse, and ρ_0 is the present-epoch mass density.

(c) According to the Zeldovich approximation the Eulerian co-moving coordinate is

$$\vec{x} = \frac{\vec{r}}{a(t)} = \vec{q} + b(t)\vec{p}(\vec{q}),$$

where $\vec{p}(\vec{q})$ is a vector field, $a(t)$ is the scale factor and $b(t)$ is the growing mode solution of the density fluctuations in linear theory, which satisfies

$$\ddot{b} + 2\frac{\dot{a}}{a}\dot{b} = 4\pi G b \frac{\rho_0}{a^3}\,,$$

and ρ_0 is the mean mass density at the present epoch.

By using the continuity equation show that

$$1 + \delta = \frac{1}{(1 - b\lambda_1)(1 - b\lambda_2)(1 - b\lambda_3)}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the deformation tensor $\frac{\partial p_i}{\partial q_j}$.

By choosing $\vec{p}(\vec{q}) = [p_1(q_1), 0, 0]$ show that in one dimension the Zeldovich approximation satisfies exactly

$$\vec{\nabla}_{\vec{r}} \cdot \ddot{\vec{r}} = -4\pi G \rho(\vec{r}, t)\,.$$