UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 1.30 to 4.30

PAPER 45

PHYSICAL COSMOLOGY

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight, parts of questions may have different weights

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Define the Cosmological Principle, and discuss the observational evidence to support it.

(b) Show that the line element of a ($k = 1$) Friedmann-Robertson-Walker universe can be written in the form

$$ds^2 = a^2(\eta)[d\eta^2 - d\chi^2 - \sin^2\chi(d\Theta^2 + \sin^2\Theta d\phi^2)]\,,$$

where $a(\eta)$ is the scale factor and the new time η is defined by $d\eta = dt/a$, and for an appropriate choice of a new radial coordinate χ. Show that for a universe with $k = 1, \Lambda = 0$, the Friedmann equation can be rewritten as

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G\rho}{3}a^4 - a^2.$$

Solve the equation for the matter dominated case and show that:

$$\begin{cases} a(\eta) = GM(1 - \cos\eta) \\ t(\eta) = GM(\eta - \sin\eta) \end{cases} \quad ,$$

where $M = \frac{4\pi}{3}\rho_0 a_0^3$.

(c) Explain what is meant by the luminosity distance d_L.

Assume that the luminosity per steradian is $P(\nu) \propto \nu^{-\alpha}$, where ν is the frequency and α is a constant. Show that the observed flux density due to a source at redshift z is

$$S(\nu_0) = P(\nu_0)(1+z)^{1-\alpha} d_L^{-2}.$$

(d) Discuss how Type Ia Supernovae can be used to constrain cosmological parameters and the equation of state.

2 (a) For small perturbations the equation of motion for a fluid element in an expanding universe ($\Lambda = 0$) is

$$\ddot{\mathbf{x}} + 2\frac{\dot{a}}{a}\dot{\mathbf{x}} = -\frac{1}{a^2}\nabla_{\mathbf{x}}\varphi,$$

where $a(t)$ is the scale factor and φ is the perturbation in the potential.

Write down the corresponding Poisson and continuity equations.

(b) Now consider the following transformations:

$$\mathbf{u} = \frac{d\mathbf{x}}{db}$$

and

$$\psi = [\frac{3}{2}H_0^2\Omega_0]^{-1}\frac{a}{b}\varphi,$$

where H_0 is the Hubble constant, Ω_0 is the present epoch density parameter, and $b(t)$ is the growing solution of the equation

$$\ddot{D} + 2\frac{\dot{a}}{a}\dot{D} - \frac{3}{2}H_0^2\Omega_0\frac{D}{a^3} = 0.$$

Show that the new Poisson, motion and continuity equations are:

$$\nabla^2\psi = \frac{\delta}{b},$$

$$\frac{d\mathbf{u}}{db} = -\frac{3}{2}\frac{\Omega(t)}{bf^2}(\mathbf{u} + \nabla\psi),$$

where $f = \frac{d\ln b}{d\ln a}$, and

$$\frac{d\delta}{db} + \nabla\cdot[(1+\delta)\mathbf{u}] = 0.$$

(c) Make use of the fact that $b(t)$ is the rate of growth of the density contrast in the linear regime to show that $\psi(\mathbf{x}, t)$ does not evolve in the linear regime.

Show that $\mathbf{u}$ is a constant in the linear regime, i.e. $\frac{d\mathbf{u}}{db} = 0$

(d) Under what condition is $\mathbf{u} = -\nabla\psi$?

Show that in this case the trajectories of particles may be described in the linear regime as:

$$\mathbf{x} = \mathbf{x}_{in} - (b - b_{in})\nabla_{\mathbf{x}_{in}}\psi_{in},$$

where $\mathbf{x}_{in}$ is the initial position of the fluid element at b_{in}.

(e) The Zeldovich approximation is the extrapolation of this equation in the mildy non-linear regime. Identifying the initial position with the Lagrangian coordinate we have

$$\mathbf{x} = \mathbf{q} - b(t)\nabla_{\mathbf{q}}\psi_{in},$$

where we have neglected b_{in} assuming it to be small in comparison with $b(t)$.

Explain how the Zeldovich approximation can be used to describe the formation of ‘pancakes’.

Discuss why at late times the Zeldovich approximation predicts pancakes which are too thick compared to pancakes seen in N-body simulations.

3 (a) Describe qualitatively and sketch a diagram of the evolution with time of the radius of a spherical proto-cluster in an Einstein-de Sitter universe, from early time to virialization.

(b) Consider a universe with a positive cosmological constant Λ. The energy per unit mass for a shell enclosing mass M is:

$$E(M) = \frac{1}{2}\dot{r}^2 - \frac{GM}{r} - \frac{1}{6}\Lambda r^2 \qquad (*)$$

Show that for a uniform sphere the potential energy at the maximum radius $r_{\rm m}$ ('turn around') is:

$$U_{\rm m} = -\frac{3}{5}\frac{GM^2}{r_{\rm m}} - \frac{1}{10}\Lambda M r_{\rm m}^2.$$

(c) The virial theorem relates the kinetic energy K to a potential energy of the form $U \ \alpha \ r^n$ by

$$K_{\rm v} = \frac{n}{2}U_{\rm v}.$$

Write down the virial theorem for the system given by equation (*).

(d) Show that the radius $r_{\rm v}$ at virialization obeys the equation:

$$2\eta\left(\frac{r_{\rm v}}{r_{\rm m}}\right)^3 - (2+\eta)\left(\frac{r_{\rm v}}{r_{\rm m}}\right) + 1 = 0,$$

where $\eta = \dfrac{\Lambda}{4\pi G\rho}$ and ρ is the density.

(e) What is the condition on η for the shell to turn around ?

(f) What is the minimal possible $\dfrac{r_{\rm v}}{r_{\rm m}}$ if $\eta > 0$, and why is it smaller than that in the case $\eta = 0$?

(g) Discuss briefly how the virial theorem can be applied to deduce the total mass of a galaxy cluster from optical and X-ray observations.

4 (a) Define the two-point correlation function $\xi(r)$, and derive the relation between $\xi(r)$ and the Fourier power spectrum of fluctuations $P(k)$.

Express your result as a one-dimensional integral over k.

(b) Derive $\xi(r)$ for a power spectrum of the form

$$P(k) = Ak \qquad \text{for} \qquad 0 \leqslant k \leqslant k_{\rm max},$$

and zero elsewhere, where A and $k_{\rm max}$ are constants.

(c) Consider a power spectrum of the form $P(k) = Ak^n$, where n is a constant.

Derive by dimensional arguments the *rms* fluctuations in the gravitational potential.

(d) Explain the assumptions made in deriving the Press-Schechter model for the mass function.

Discuss the time evolution of the mass function in a hierarchical model of structure formation.

Explain how the observed temperature function of galaxy clusters can be used to constrain the amplitude of the density fluctuations.