UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 9 to 11

PAPER 72

PHASE TRANSITIONS AND COLLECTIVE PHENOMENA

*Attempt **TWO** questions. The questions are of equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain the concepts of *Spontaneous Symmetry Breaking* and *Goldstone modes* in statistical mechanics.

The low-energy properties of a classical d-dimensional XY-Ferromagnet are described by the Ginzburg-Landau Hamiltonian

$$\beta H = \frac{\bar{K}}{2} \int d^d\mathbf{x} \, (\nabla\theta)^2 \ ,$$

where the corresponding two-component magnetisation field $\mathbf{m}(\mathbf{x}) = \bar{m}(\cos\theta(\mathbf{x}), \sin\theta(\mathbf{x}))$ is assumed to be constant in magnitude.

(a) Taking the fluctuations of the magnetisation field to be small, i.e. $\theta(\mathbf{x}) \ll 2\pi$, use the rules of Gaussian functional integration to show that the correlation function takes the form

$$\langle\theta(\mathbf{x})\theta(0)\rangle = -\frac{|\mathbf{x}|^{2-d}}{(2-d)S_d\bar{K}} + \text{const.}\,,$$

where S_d denotes the d-dimensional solid angle.

(b) Using this result, show that

$$\lim_{|\mathbf{x}|\to\infty} \langle\mathbf{m}(\mathbf{x})\cdot\mathbf{m}(0)\rangle = \begin{cases} m_0^2 & d > 2, \\ 0 & d \le 2, \end{cases}$$

where m_0 denotes some non-zero constant. Comment on the implications of this result for the nature of long-range order in low dimensions.

(c) Explain *qualitatively* how the topological character of the field $\theta(\mathbf{x})$ influences the behaviour of the XY-spin system in precisely two-dimensions.

2 Describe the conceptual basis of the *scaling hypothesis* as applied to phenomenology of a second order critical point.

Close to the critical point of a classical Ferromagnet, the singular part of the free energy assumes the homogeneous form

$$f(t,h) = t^{2-\alpha} g_f\left(\frac{h}{t^{\Delta}}\right),$$

where $t = (T - T_c)/T_c$ represents the reduced temperature, and h denotes the dimensionless magnetic field.

(a) Starting with the expression for the free energy density, show that the magnetisation assumes a homogeneous form. From this result, determine the relation between the scaling exponents of the magnetisation $m(t, h = 0) \sim t^{\beta}$ and $m(t = 0, h) \sim h^{1/\delta}$ and the exponents α and Δ.

(b) Using the expression for the magnetisation, obtain the relation between the scaling exponent γ of the susceptibility $\chi(t) \sim t^{-\gamma}$ and the exponents α and Δ.

(c) According to the hyperscaling hypothesis, close to the critical point, the correlation length assumes the homogeneous form

$$\xi(t,h) = t^{-\nu} g_{\xi}\left(\frac{h}{t^{\Delta}}\right).$$

Explain why this result is compatible with the hyperscaling identity $d\nu = 2 - \alpha$.

(d) According to the scaling hypothesis, the correlation function takes the form

$$\langle m(\mathbf{x}) m(0)\rangle = \frac{1}{|\mathbf{x}|^{d-2+\eta}} g\left(\frac{|\mathbf{x}|}{\xi(t,h)}\right).$$

From this result, obtain the susceptibility and prove the exponent identity $\gamma = (2-\eta)\nu$.

3 Write notes on **one** of the following topics:

(a) the Ginzburg-Landau theory; mean-field and fluctuation phenomena;

(b) the Ginzburg criterion;

(c) conceptual foundations of the renormalisation group.

4 The one-dimensional lattice Ising ferromagnet is described by the microscopic Hamiltonian

$$\beta H = -\sum_{ij} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i,$$

where h denotes the magnetic field and the exchange interaction varies with separation between sites i and j as $J_{ij} = Je^{-\kappa|i-j|}$ with $\kappa \ll 1$.

(a) By employing an appropriate Hubbard-Stratonovich transformation, show that the partition function is given by

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\sum_{ij} m_i[J^{-1}]_{ij}m_j + \sum_i \ln(2\cosh(2m_i + h))\right],$$

where C represents some unspecified constant.

(b) For the long-ranged model defined above, show that

$$\mathcal{Z} = C\int_{-\infty}^{\infty}\prod_k dm_k \exp\left[-\sum_j \left(\frac{1}{2J\sinh\kappa}(m_j - m_{j+1})^2 + U(m_j)\right)\right],$$

where $U(m) = \tanh(\kappa/2)m^2/J - \ln[2\cosh(2m+h)]$.

(c) Taking the continuum limit, show that the classical partition function is isomorphic to the quantum transition probability of a particle in a double well potential. Moreover, show that the external magnetic field h generates an asymmetry of the potential.