UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 1.30 to 3.30

PAPER 48

PERTURBATION METHODS

ALL *questions may be attempted,*
full marks may be obtained by substantially complete answers to ***TWO*** *questions*

There are ***four*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 **(a)** The integral $I(\lambda)$ is defined by

$$I(\lambda) = \int_1^\infty \frac{1}{x^2} \exp\left(-\lambda \exp(-x)\right) \, \mathrm{d}x \,.$$

Find the asymptotic expansion for $I(\lambda)$ as $\lambda \to \infty$ correct to, and including, terms that are $\mathcal{O}\big((\ln \lambda)^{-2} \big)$.

(b) The integral $\mathcal{I}(\sigma)$ is defined by

$$\mathcal{I}(\sigma) = \int_0^{2\pi} \exp\left(-\sigma x(1 - \cos x)\right) \, \mathrm{d}x \,.$$

Find the the first two terms of the asymptotic expansion for $\mathcal{I}(\sigma)$ as $\sigma \to \infty$.

It may help to recall that

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) \, \mathrm{d}t \,,$$

and that

$$\int_0^\infty \ln(t) \exp(-t) \, \mathrm{d}t = -\gamma \,,$$

where γ *is Euler's constant.*

2 The function $g(s;t,a,\sigma)$ is defined by

$$g(s;t,a,\sigma) = -\frac{1}{a}\,\mathrm{e}^{\mathrm{i}as}\int_0^\infty \frac{x^{s+\sigma-1}\mathrm{e}^{-tx}}{\mathrm{i}a+\ln x}\,\mathrm{d}x\,,$$

where $s \geq 0$, $t > 0$, $a > 0$ and $\sigma > 0$. Let

$$F(t,a,\sigma) = \mathrm{Im}(g(0;t,a,\sigma))\,.$$

(a) Show that

$$|g(s;t,a,\sigma)| \leq \frac{1}{a}\int_0^\infty \frac{x^{s+\sigma-1}\mathrm{e}^{-tx}}{(a^2+\ln^2 x)^{\frac{1}{2}}}\,\mathrm{d}x\,.$$

Also, by considering $g_s(s;t,a,\sigma)$ or otherwise, show that

$$g(0;t,a,\sigma) = \frac{1}{a}\int_0^s \mathrm{e}^{\mathrm{i}ax}\,\frac{\Gamma(x+\sigma)}{t^{x+\sigma}}\,\mathrm{d}x + g(s;t,a,\sigma)\,.$$

(b) Using the above two results deduce that as $t \to \infty$

$$F(t,a,\sigma) \sim t^{-\sigma}\sum_{n=0}^{\infty}\beta_n(\ln t)^{-n-1}\,,$$

where the $\beta_n(a,\sigma)$ for $n = 0, 1, \ldots$ are defined through the generating function

$$\frac{\sin ax}{a}\,\Gamma(\sigma+x) = \sum_{n=0}^{\infty}\beta_n\,\frac{x^n}{n!}\,.$$

(c) State the asymptotic behaviour as $t \to \infty$ of Ramanujan's function

$$N(t) = \int_0^\infty \frac{\mathrm{e}^{-tx}}{x(\pi^2+\ln^2 x)}\,\mathrm{d}x\,.$$

After noting the range of parameter values for which $g(s;t,a,\sigma)$ is defined, briefly justify your answer.

3 For $1 \leq x < \infty$, the function $y(x)$ satisfies the ordinary differential equation

$$x^2 y'' - (1 - \varepsilon(1 + y'))\, xy' - 2\varepsilon y = 0\,,$$

where $\varepsilon \ll 1$ is a small positive constant.

(a) If

$$y(1) = 1\,, \qquad y'(1) = 1\,,$$

find the leading-order solution for $y(x)$ for $1 \leq x < \infty$.

(b) Suppose that instead

$$y(1) = 1\,, \qquad y'(1) = 0\,.$$

Again find the leading-order solution for $y(x)$ for $1 \leq x < \infty$.

Hint. x might not always be the ideal variable.

4 For $t \geq 0$, the function $u(t)$ satisfies the ordinary differential equation

$$\ddot{u} + 4\mathrm{e}^{-2\varepsilon t}\, u = 8\mathrm{e}^{-2t}\,,$$

where $\varepsilon \ll 1$ is a small positive constant. At $t = 0$, u satisfies the initial conditions

$$u(0) = 1\,, \qquad u'(0) = -1\,.$$

Find the WKB solution correct to $O(\varepsilon)$ that is valid for times such that $\varepsilon t = O(1)$.

Is the WKB solution found above valid if $\varepsilon \mathrm{e}^{\varepsilon t} = O(1)$? If not propose a rescaling about such times and derive, but do not solve, the governing equation for the leading-order solution, and give matching conditions.