UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2003 9 to 11

PAPER 8

ORDINARY DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

The first three questions relate to the 2×2 system of first order ODE's

$$\frac{dy}{d\lambda} = A(\lambda)y \tag{1}$$

with $A(\lambda) = A_0\lambda^2 + A_1\lambda + A_2$.

1 What is the order of the pole at ∞?

What is the Poincaré rank of the singularity $\lambda = \infty$?

If $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ determine the Stokes rays.

How many true solutions of given asymptotic behaviour are needed to cover a sector of opening $2\pi + \epsilon$ at ∞? ($\epsilon > 0$, small)

Using the fact that $\lambda = \infty$ is the only singularity of the system (1), show that the Stokes matrices $S_1, S_2, \ldots, S_6$ satisfy the following relation

$$S_6S_5S_4S_3S_2S_1 = 1$$

2 If

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & u(z) \\ \frac{-zw(z)}{u(z)} & 0 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} w(z) + z/2 & u'(z) \\ \frac{2u'(z)w(z)}{[u(z)]^2} & -w(z) - z/2 \end{pmatrix} \tag{2}$$

determine the formal solution at ∞ of (1).

[*Hint: determine the formal diagonalization*

$$\Lambda(\lambda) = A_0\lambda^2 + \Lambda_1\lambda + \Lambda_2 + \Lambda_3/\lambda + \mathcal{O}(1/\lambda)$$

of $A(\lambda)$ *by the gauge formula*

$$A(\lambda)G(\lambda) = G'(\lambda) + G(\lambda)\Lambda(\lambda)$$

where $G(\lambda) = 1 + \frac{G_1}{\lambda} + \frac{G_2}{\lambda^2} + \cdots$

Then

$$Y_f = \left(1 + \frac{G_1}{\lambda} + \frac{G_2}{\lambda} + \mathcal{O}(\frac{1}{\lambda^3})\right) \lambda^{\Lambda_3} \exp\left(\frac{A_0\lambda^3}{3} + \frac{\Lambda_1\lambda^2}{2} + \Lambda_2\lambda\right).]$$

By studying the analytic properties of the function $B(\lambda) = \left(\frac{d}{dz}y\right)\ y^{-1}$ where $y(\lambda)$ is a fundamental matrix of (1), show that the isomonodromic deformation equation of y as a function of z is

$$\frac{dy}{dz} = \begin{pmatrix} \lambda/2 & u(z)/2 \\ -\frac{w(z)}{u(z)} & -\lambda/2 \end{pmatrix} y$$

3 Consider the two systems

$$\begin{cases} \frac{\partial}{\partial\lambda}y = & A(\lambda)y \\ \frac{\partial}{\partial z}y = & B(\lambda)y \end{cases}$$

where $A(\lambda, z) = A_0\lambda^2 + A_1\lambda + A_2$ with A_0, A_1, A_2 given in (2) and

$$B(\lambda, z) = \begin{pmatrix} \lambda/2 & u(z)/2 \\ -\frac{w(z)}{u(z)} & -\lambda/2 \end{pmatrix}$$

Show that the "compatability condition"

$$\frac{\partial^2}{\partial\lambda\partial t}y = \frac{\partial^2}{\partial z\partial\lambda}y$$

leads to the ODE:

$$w(z) = u'(z), \quad u''(z) + Cu^3(z) + \frac{t}{z}u(z) = 0\,,$$

where C is an arbitrary constant.

4 State and explain the Painlevé Property.

Show that the following Riccati differential equation satisfies the Painlevé property

$$\frac{dw}{dz} \;=\; z(z+1)w^2 \;-\; \frac{3}{z+1}w \;-\; \frac{1}{z^2(z+1)^2}\,.$$