UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 1.30 to 4.30

PAPER 71

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt **THREE** questions from **Section A** and attempt **ONE** question from **Section B**.*

*There are **SEVEN** questions in total.*

*Each question from **Section B carries twice the weight** of a question from **Section A**.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Given the ODE system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $t \geqslant 0$, $\mathbf{y}(0) = \mathbf{y}_0$, we let

$$\mathbf{g}(\mathbf{y}) = \frac{\partial \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}} \mathbf{f}(\mathbf{y})$$

and consider the two-step, two-derivative method

$$\mathbf{y}_{n+1} - \alpha h \mathbf{f}(\mathbf{y}_{n+1}) + \beta h^2 \mathbf{g}(\mathbf{y}_{n+1}) = \gamma h \mathbf{f}(\mathbf{y}_n) + \mathbf{y}_{n-1} + \alpha h \mathbf{f}(\mathbf{y}_{n-1}) + \beta h^2 \mathbf{g}(\mathbf{y}_{n-1}).$$

(a) Determine real numbers α, β, γ so that this method is of order $p = 5$.

(b) Prove directly (i.e., without recourse to the second Dahlquist barrier) that the above method is not A-stable.

[You might use without a proof the Cohn–Schur criterion: both zeros of the quadratic equation $Aw^2 + Bw + C = 0$, where A, B, C are complex and $A \neq 0$, are in $|w| \leqslant 1$ iff $|A| \geqslant |C|$, $(|A|^2 - |C|^2)^2 \geqslant |A\bar{B} - B\bar{C}|^2$ and, if the latter inequality is an equality, $|B| \leqslant 2|A|$.]

2 The linear ODE $\mathbf{y}' = P\mathbf{y}$, where the matrix P is skew-symmetric, is solved by a ν-stage Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^\top \end{array}$$

(a) Prove that $\|\mathbf{y}(t)\| \equiv \|\mathbf{y}_0\|$, $t \geqslant 0$, where $\|\cdot\|$ is the Euclidean norm.

(b) Show that the above property is conserved by the Runge–Kutta method, i.e. $\|\mathbf{y}_n\| \equiv \|\mathbf{y}_0\|$, $n \geqslant 0$, if it is true in every step n that

$$2 \sum_{m=1}^{\nu} b_m \mathbf{y}_n^\top \mathbf{k}_m + h \left\| \sum_{m=1}^{\nu} b_m \mathbf{k}_m \right\|^2 = 0.$$

Here $\mathbf{k}_1, \ldots, \mathbf{k}_\nu$ are the stages of the RK method.

(c) Deduce that a sufficient condition for $\|\mathbf{y}_n\| \equiv \|\mathbf{y}_0\|$, $n \geqslant 0$, is that $M = O$, in other words that

$$b_m a_{m,j} + b_j a_{j,m} - b_m b_j = 0, \qquad m, j = 1, \ldots, \nu.$$

3 The Cauchy problem for advection equation $u_t = u_x$ is solved by the finite-difference method

$$\frac{1}{2}\mu(1+\mu)u_{m-1}^{n+1} + (1+\mu)(2-\mu)u_m^{n+1} + \frac{1}{2}(1-\mu)(2-\mu)u_{m+1}^{n+1} = (2-\mu)u_m^n + (1+\mu)u_{m+1}^n,$$

where μ is the Courant number.

(a) Prove that the method is of order 3.

(b) Determine the range of μ for which the method is stable.

4 We are given the equation

$$u_t = \nabla^2 u + \kappa u, \qquad 0 \leqslant x, y \leqslant 1, \quad t \geqslant 0,$$

together with an L_2 initial condition at $t = 0$ and zero boundary conditions. κ is a real constant.

(a) Prove that for $\kappa \leqslant 2\pi^2$ the equation is well posed and that $\kappa < 2\pi^2$ implies that $\lim_{t\to\infty} u(x, y, t) = 0$ for all $x, y \in [0, 1]$.

(b) The equation is semidiscretized by replacing the Laplacian with the five-point formula. Determine the range of κ for which the method is stable.

5 Let

$$\mathcal{L} = -\frac{d}{dx}p(x)\frac{d}{dx} + q(x), \qquad 0 \leqslant x \leqslant 1,$$

be a differential operator. Here $p(x) > 0$ and $q(x) \geqslant 0$ are differentiable functions.

(a) Carefully quoting all necessary theorems and proving that all required conditions are satisfied, prove that the differential equation $\mathcal{L}u = f$ is the Euler–Lagrange equation of the variational problem

$$I(v) = \langle \mathcal{L}v, v \rangle - 2\langle f, v \rangle,$$

where $\langle f, g \rangle$ is the standard L_2 inner product and the function v resides in a suitable function space which you should identify.

(b) Describe and justify a Galerkin method for the equation $\mathcal{L}u = f$, given in tandem with the boundary conditions $u(0) = u(1) = 0$.

SECTION B

6 Write an essay on collocation methods for ODEs, their interpretation as a specific type of Runge–Kutta methods and their order analysis.

7 Write an essay on Fourier stability analysis for finite difference methods for PDEs of evolution. You should comment on the suitability and different treatment of different kinds of boundary conditions.

END OF PAPER