UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 9 to 11

PAPER 78

NON-NEWTONIAN FLUID DYNAMICS

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 What is meant by saying that the second rank tensor $\mathbf{J}(\mathbf{x}, t)$ is an *objective* quantity?

Show from first principles that the stress $\boldsymbol{\sigma}$ and the rate of strain $\mathbf{E}$ are both objective quantities. Show also that if $\mathbf{J}$ is objective, then so is the (non-standard) time derivative

$$\overset{\diamond}{\mathbf{J}} \equiv \frac{D\mathbf{J}}{Dt} + \boldsymbol{\Omega} \cdot \mathbf{J} - \mathbf{J} \cdot \boldsymbol{\Omega} - \alpha(\mathbf{E} \cdot \mathbf{J} + \mathbf{J} \cdot \mathbf{E})$$

where $\boldsymbol{\Omega}$ is the vorticity tensor and α is a constant.

For a Johnson-Segalman fluid the stress is given as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \alpha G_0 \mathbf{A} + 2\mu_0 \mathbf{E},$$

$$\overset{\diamond}{\mathbf{A}} = -\frac{1}{\tau}(\mathbf{A} - \mathbf{I}),$$

with G_0, μ_0 and τ all positive constants, and $|\alpha| < 1$. Find the steady extensional viscosity and comment on your result.

Find also the viscometric functions $\mu(\dot{\gamma}), \psi_1(\dot{\gamma})$ and $\psi_2(\dot{\gamma})$ in simple shear. Sketch the possible behaviours of the shear stress $\mu(\dot{\gamma})\dot{\gamma}$, and comment on the physical significance of the sketch for the case in which $\mu_0/\alpha^2 G_0 \tau$ is small.

2 Under what circumstances does a generalised Newtonian fluid having constitutive equation

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu(\dot{\gamma})\mathbf{E} \quad \text{with} \quad \dot{\gamma} = (2\mathbf{E} : \mathbf{E})^{1/2}$$

provide a good approximation for the behaviour of a viscoelastic fluid?

Specify *two* non-Newtonian phenomena that this equation *cannot* explain.

A power-law fluid with

$$\mu(\dot{\gamma}) = k|\dot{\gamma}|^{n-1}$$

occupies the channel between the planes $y = \pm h$. The planes are caused to move steadily in the $x-$direction with velocities $\pm U$ respectively, and $U > 0$. In addition, a constant pressure gradient $-G$ is applied to the fluid in the $x-$direction. Taking the shear stress on the wall $y = h$ to be $Gh(\alpha - 1)$, find the velocity profile in terms of α, and give an equation for α. Find α explicitly for a Newtonian fluid.

In circumstances where $Gh \ll k(U/h)^n$, find a simplified expression for the velocity profile and deduce the volume flux Q through the channel. Sketch this flow profile.

How may this result be extended so as to apply to any generalised Newtonian fluid? What is the corresponding expression for Q?

3 Write down an expression for the stress in a *linear viscoelastic fluid* in terms of its relaxation modulus $G(s)$ and specify the circumstances in which this equation is valid.

Deduce the zero shear rate viscosity, μ, of such a fluid.

The constitutive equation of an Oldroyd fluid is given as

$$\boldsymbol{\sigma} = -p\mathbf{I} + G_0\mathbf{A} + 2\mu_0\mathbf{E},$$

$$\frac{D\mathbf{A}}{Dt} - \nabla\mathbf{v}^{\mathrm{T}} \cdot \mathbf{A} - \mathbf{A} \cdot \nabla\mathbf{v} = -\frac{1}{\tau}(\mathbf{A} - \mathbf{I}),$$

with G_0, μ_0 and τ all positive constants. Find the relaxation modulus of the fluid and hence μ.

This fluid is placed in a cone-and-plate viscometer. The angle between the cone and the plate is $\alpha \ll 1$; the radius of the stationary plate is a. Inertia is negligible. For times $t < 0$ a steady torque T is applied to the cone. Find, in the linear viscoelastic limit, the steady angular velocity Ω of the cone.

For times $t > 0$ a steady torque $-T$ is applied to the cone. Find its induced angular velocity $\Omega(t)$ for all $t > 0$.

[*Hint: the Laplace transform of* $e^{-\beta t}$ *is* $1/(\beta + p)$ *for* $\beta \geq 0$.]