UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 31 May 2002 1.30 to 3.30

PAPER 53

NON-NEWTONIAN FLUID DYNAMICS

*Attempt **TWO** questions*

*There are **three** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what is meant by a *viscometric flow* of a simple fluid. Explain *briefly* why the stress in such a flow is given as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{A}_1 + (\psi_1 + \psi_2)\mathbf{A}_1^2 - \tfrac{1}{2}\psi_1\mathbf{A}_2,$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivlin-Ericksen tensors for the flow, and μ, ψ_1 and ψ_2 are functions of $\dot{\gamma} = [\frac{1}{2}\mathbf{A}_1 : \mathbf{A}_1]^{1/2}$. [*Results from matrix algebra may be quoted without proof.*] Explain briefly the physical significance of the functions μ, ψ_1 and ψ_2.

Give a scaling argument to estimate for a lubrication flow the circumstances in which the terms involving ψ_1 and ψ_2 may be neglected.

For a power-law fluid the viscosity is given as

$$\mu = k|\dot{\gamma}|^{n-1},$$

where k and n are positive constants. Such a fluid, having density ρ, coats the outside of a circular cylinder of radius a that rotates steadily about a horizontal axis with angular velocity Ω. Gravity acts vertically, and the shear stress at the free surface is zero. Let $h(\theta)$ be the steady thickness of the fluid layer, with $\theta = 0$ horizontal. Neglecting normal stresses, use lubrication theory to find the shear stress throughout the layer, and determine the total fluid flux, Q, in the layer as

$$Q = \Omega a h \mp \frac{n}{2n+1}\left(\frac{\rho g|\cos\theta|}{k}\right)^{1/n} h^{(2n+1)/n},$$

where the sign is chosen according as $\cos\theta$ is positive or negative.

Hence show that there is a maximum possible steady flux, and find it.

2 Under what circumstances is the behaviour of a simple fluid approximated by that of a linear viscoelastic fluid?

The fluid stress in this limit is given as

$$\boldsymbol{\sigma}(\mathbf{x},t) = -p\mathbf{I} + 2\int_{-\infty}^{t} G(t-t')\mathbf{E}(\mathbf{x},t')\,dt',$$

where G is the relaxation modulus. Define the terms *complex viscosity* and *creep* and explain how each of these quantities may be determined from G.

A Maxwell fluid having density ρ and relaxation modulus

$$G(t) = G_0 e^{-t/\tau}$$

occupies the planar region $y > 0$ bounded by a rigid wall along $y = 0$. For times $t < 0$ the fluid is at rest. For times $t \geq 0$ the wall is moved in the $x-$direction with constant velocity U. Use Laplace transforms or otherwise to obtain the velocity in the fluid for times $t > 0$ in the form

$$u(y,t) = \frac{U}{2\pi i}\int \frac{dp}{p}\exp\{-[p(1+p\tau)\rho/G_0\tau]^{1/2}y + pt\},$$

where the contour of integration should be specified.

By taking appropriate limits of the material parameters, find $u(y,t)$ for both a Newtonian viscous fluid and an elastic solid. [*The Laplace transform of* $\mathrm{erfc}(k/2\sqrt{t})$ *is* $\frac{1}{p}\exp(-k\sqrt{p})$.]

Is it legitimate to apply linear viscoelasticity to this problem? Explain your answer briefly.

3 The pom-pom model for an incompressible entangled polymer melt gives the stress $\boldsymbol{\sigma}(t)$ in terms of a molecular strain, $\lambda(t)$, and a molecular orientation, $\mathbf{B}(t)$, as

$$\boldsymbol{\sigma} = G\lambda^2\mathbf{B} - p\mathbf{I},$$

with

$$\mathbf{B} = \mathbf{A}/\mathrm{trace}(\mathbf{A}),$$

where $\mathbf{A}$ evolves according to

$$\frac{D\mathbf{A}}{Dt} = \mathbf{A}.\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T.\mathbf{A} - \frac{1}{\tau_1}(\mathbf{A} - \mathbf{I})$$

and λ evolves according to

$$\frac{D\lambda}{Dt} = \lambda\mathbf{B} : \boldsymbol{\nabla}\mathbf{v} - \frac{1}{\tau_2}(\lambda - 1).$$

In these equations, G, τ_1 and τ_2 are positive constants.

(a) Does this constitutive equation describe a simple fluid? Briefly justify your answer.

(b) What are $\mathbf{B}$ and λ for a state of rest? By expanding about this state, obtain the stress relaxation modulus of the fluid and the second-order-fluid constants.

(c) Consider finally the imposition of a steady, uniaxial extensional flow withi principal rate of strain $\dot{\gamma}$, starting from rest at $t = 0$. Show that if $\dot{\gamma}\tau_1 < \frac{1}{2}$ then $\mathbf{A}(t)$ tends to a finite limit as $t \to \infty$, and that if $\dot{\gamma}\tau_1 > \frac{1}{2}$ then $\mathbf{A}(t)$ increases without bound. Show that in either case $\mathbf{B}(t)$ tends to a finite limit as $t \to \infty$ to be determined.

Deduce that $\lambda(t)$ remains bounded as $t \to \infty$ only if $\dot{\gamma}$ is less than a critical value $\dot{\gamma}_c$ which you need not determine. Find an expression for the steady extensional viscosity of the fluid when both $\dot{\gamma} < \dot{\gamma}_c$ and $\dot{\gamma}\tau_1 < \frac{1}{2}$.