UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 9 to 11

PAPER 49

NON-NEWTONIAN FLUID MECHANICS

*Attempt any **TWO** questions. The questions carry equal weight.*

The following definitions may be quoted:

$$\overset{\triangledown}{J} \equiv \frac{D\boldsymbol{J}}{Dt} - \boldsymbol{J}\cdot\nabla\boldsymbol{v} - \nabla\boldsymbol{v}^T\cdot\boldsymbol{J}\ ,$$
$$\overset{\circ}{J} \equiv \overset{\triangledown}{J} + \boldsymbol{J}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{J}\ ,$$
$$\overset{\triangle}{J} \equiv \overset{\circ}{J} + \boldsymbol{J}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{J}\ .$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The deviatoric stress in a linear viscoelastic fluid has the form

$$\boldsymbol{\sigma}'(\boldsymbol{x}, t) = 2\int_0^\infty G(s)\boldsymbol{E}(\boldsymbol{x}, t-s)ds \ .$$

Explain briefly the circumstances in which this equation is expected to approximate the behaviour of a viscoelastic fluid (formal justification is *not* required). Explain also why $G(s)$ is called the *relaxation modulus* of the fluid.

Define the *complex viscosity* $\hat{\mu}(\omega)$ of such a fluid and explain the physical significance of its real and imaginary parts. Find $\hat{\mu}(\omega)$ for a Maxwell fluid such that

$$G(s) = G_0 e^{-s/\tau} \ .$$

A Maxwell fluid of density ρ occupies a channel between the fixed planes $y = \pm h$. An oscillatory pressure gradient $\Delta p e^{i\omega t}$ is applied to the fluid in the x-direction. Assuming that the flow is unidirectional, determine the volume flux per unit length in the z-direction, $Qe^{i\omega t}$ as

$$Q = -\frac{2\Delta p h}{i\omega\rho}\left[1 - \frac{\tanh z}{z}\right] \ , \text{ where}$$
$$z = \left[i\omega\rho h^2(1 + i\omega\tau)/G_0\tau\right]^{\frac{1}{2}} \ .$$

Obtain the limiting forms of Q as $\omega \to 0$ and $\omega \to \infty$ and comment on your results.

Suppose finally that $\omega\tau \gg 1$ but h is sufficiently small that $\rho(\omega h)^2/G_0 \ll 1$. Find Q in this limit and comment.

2 A rigid sphere rotates slowly and steadily in an unbounded fluid. Sketch the secondary flow streamlines that you would expect as a result of weak non-Newtonian effects. Briefly describe *two* other non-Newtonian flow phenomena that result from the same mechanism.

Explain briefly (detailed derivations are *not* required) why the constitutive equation for a weakly non-Newtonian fluid is given by the second-order-fluid result

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{E} + 4(\psi_1 + \psi_2)\boldsymbol{E}^2 - \psi_1 \overset{\triangle}{\mathbf{E}} \ .$$

In what circumstances would you use this equation? Calculate the stress for a steady simple shear flow.

It may be shown that for any Stokes flow $\boldsymbol{v}_N(x)$,

$$\nabla_\wedge(\nabla . \overset{\circ}{\mathbf{E}}_N) = 0 \ .$$

Deduce that, for an inertialess flow of a second-order-fluid having $\psi_1 + 2\psi_2 = 0$, if the velocity $\boldsymbol{v}$ is prescribed on the bounding surface, then $\boldsymbol{v}(\boldsymbol{x}) \equiv \boldsymbol{v}_N(\boldsymbol{x})$ throughout the fluid.

Deduce the same result for any planar flow of an *arbitrary* second-order-fluid.

3 A PTT fluid, used to model the flow of polymer melts, has the constitutive equation

$$\overset{\nabla}{\mathbf{A}} + \frac{1}{\tau}(1 + \alpha \mathrm{tr}\boldsymbol{A})\boldsymbol{A} = 2\boldsymbol{E}$$
$$\boldsymbol{\sigma} = -p\boldsymbol{I} + G_0\boldsymbol{A}$$

where τ, α and G_0 are constants, and $\mathrm{tr}\boldsymbol{A} \equiv a_{11} + a_{22} + a_{33}$.

(a) Simplify the equation in the linear viscoelastic limit and determine the relaxation modulus $G(s)$.

(b) Show that in a steady simple shear flow $\boldsymbol{v} = \dot{\gamma}(y, 0, 0)$,

$$\dot{\gamma}\tau = (1 + 2\alpha a_{12}^2)a_{12} \ .$$

Use this result to determine the flow profile $w(r)$ for flow in a circular cylinder of radius R under an axial pressure gradient Δp.

[*The axial equation of motion in cylindrical co-ordinates is*

$$\frac{1}{r}\frac{d}{dr}(r\sigma_{rz}) = \frac{dp}{dz} \ .$$]

(c) Consider finally a steady uniaxial extensional flow with principal strain rate $\dot{\gamma} > 0$ in the x-direction. Show that in such a flow $T = \alpha \mathrm{tr}\boldsymbol{A} - 2\dot{\gamma}\tau + 1$ satisfies

$$T(T + 2\dot{\gamma}\tau - 1)(T + 3\dot{\gamma}\tau) = 6\alpha(\dot{\gamma}\tau)^2 \ ,$$

and deduce the extensional viscosity of the fluid in the limits $\dot{\gamma}\tau \to 0$ and $\dot{\gamma}\tau \to \infty$.