MATHEMATICAL TRIPOS Part III

Friday 30 May 2008 1.30 to 4.30

PAPER 8

NON-COMMUTATIVE ALGEBRAS

Attempt no more than **THREE** *questions.*

There are **FIVE** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let K be the kernel of the canonical map $SL_2(\mathbb{Z}_p) \longrightarrow SL_2(\mathbb{F}_p)$. Define the Iwasawa algebra Ω_K and show that it is a left Noetherian domain. Sketch why Ω_K has a classical ring of quotients.
(You may assume that for an appropriate negative filtration of Ω_K, the associated graded ring is a commutative polynomial algebra.)

2 Let R be a left Noetherian ring. Define what is meant by a uniform left R-module and by an essential extension.
Show that any non-zero finitely generated left R-module M is an essential extension of a direct sum of finitely many uniform submodules V_i.
Define the injective hull $E(M)$ of M, showing that it is unique up to isomorphism and that it is isomorphic to the direct sum of the $E(V_i)$. Show that the endomorphism ring $\mathrm{End}_R(E(V_i))$ has a unique maximal ideal J such that $\mathrm{End}_R(E(V_i))/J$ is a division ring. Describe $\mathrm{End}_R(E(M))$.
Let $R = M = A_1$, the first Weyl algebra. What are $E(M)$ and $\mathrm{End}_R(E(M))$?

3 Write an essay about finitely generated modules M of the second Weyl algebra A_2.
You should include an explanation of why $d(M)$ can take the values 2, 3 or 4 for non-zero M, and illustrate by example that each of these values is achieved.

4 Define the Gelfand-Kirillov and Krull dimensions of a finitely generated left Noetherian algebra R. Show that the Gelfand-Kirillov dimension of the enveloping algebra of the Lie algebra $sl_2(\mathbb{C})$ is 3, but that its Krull dimension is 2.

5 Define what is meant by a Hopf algebra, and give examples of
(i) a commutative, non-cocommutative Hopf algebra, and
(ii) a non-commutative, cocommutative Hopf algebra.
Explain, using an example, how R-matrices may be used in the construction of non-commutative, non-cocommutative Hopf algebras.

END OF PAPER