

PAPER 27

MORSE THEORY

*Attempt **FOUR** questions.*

*There are **six** questions in total.*

The questions carry equal weight.

You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.

1 Let X be a cell complex, let $X_n \subseteq X$ denote the n -skeleton (the union of cells of dimension no more than n) and $f : S^n \rightarrow X$ a continuous map from the n -sphere to X . Show carefully that f is homotopic to a continuous map $g : S^n \rightarrow X_n$. You should clearly state any theorems that you use in the process.

2 State and prove the Morse Lemma.

3 Define the notions of Morse function, gradient-like vector field, and the Morse-Smale transversality condition. Explain (without detailed proof) how the latter allows you to compute the integral homology of a manifold, and illustrate it for the height function on the Klein bottle.

4 Let Σ_g be the closed orientable surface of genus g . Prove that any Morse function on $S^1 \times \Sigma_g$ must have at least $4g + 4$ critical points, and describe a Morse function which achieves this minimum.

5 Recall that the Grassmannian $\mathbf{Gr}(k, n)$ of complex k -dimensional subspaces of \mathbf{C}^n is identifiable with the manifold of rank k projection matrices of size $n \times n$. Choose a diagonal matrix $C = \text{diag}(c_1, \dots, c_n)$ with $c_1 < \dots < c_n$, and consider the function $P \mapsto f(P) = \text{Tr}(CP)$. Determine the critical points and their indices, prove that this is a Morse function and hence describe the homology of $Gr(k, n)$. If it helps, you may specialize to concrete values, such as $Gr(2, 4)$, but you may *not* choose $k = 0, 1, n - 1, n$.

[You may assume that $\mathbf{Gr}(k, n)$ is a submanifold of the space of complex $n \times n$ matrices, and that its tangent space at P is spanned by the infinitesimal conjugation action of unitary matrices.]

6 Explain what is meant by attaching a handle, with reference to framed links in \mathbf{R}^3 , and explain how you can construct the four-manifold $S^2 \times S^2$ in this manner.