MATHEMATICAL TRIPOS Part III

Friday 6 June 2008 9.00 to 12.00

PAPER 4

MODULAR REPRESENTATION THEORY OF FINITE GROUPS

Attempt no more than **THREE** *questions.*

There are **SIX** *questions in total.*

The questions carry equal weight.

In this paper G is a finite group. In the usual notation $(K, \mathfrak{O}, k)$ is a splitting p-modular system for G where p, the characteristic of k, is a prime dividing $|G|$. Throughout $R \in \{\mathfrak{O}, k\}$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a central, primitive idempotent in a commutative ring, and say what it means to say that two idempotents are orthogonal. What is a block of a group algebra RG? You should ensure that you demonstrate the equivalence of the definition in terms of ideals and the definition in terms of idempotents. Prove that a block decomposition is unique up to ordering.

Take $R = k$. What do we mean by a principal indecomposable module P for kG? Prove that P has a unique maximal submodule $J(P)$. Define the Cartan matrix $\underline{C}_G$ of kG.

Let $1 \neq N$ be a normal p-subgroup of G and set $\bar{G} = G/N$. Let $\tau : kG \to k\bar{G}$ be the algebra homomorphism induced by the canonical homomorphism $G \to \bar{G}$.

(a) By considering the augmentation ideal of kN show that $\ker \tau$ is a nilpotent ideal in kG.

(b) Let $\mathfrak{C}$ be a conjugacy class in G such that $\mathfrak{C} \cap C_G(N) = \emptyset$. By considering orbits of N on $\mathfrak{C}$, and letting $[\mathfrak{C}]$ denote the class sum, prove that $[\mathfrak{C}] \in \ker \tau$.

Suppose in addition that $G = NC_G(N)$. Using Idempotent Refinement, or otherwise, deduce that τ induces a one-to-one correspondence between block idempotents of kG and those of $k\bar{G}$.

Using induction on $|N|$, or otherwise, show that $\underline{C}_G = |N|\underline{C}_{\bar{G}}$.

2 Let R be a commutative ring of coefficients such that the Krull-Schmidt theorem holds for finitely-generated RG-modules. If M is an indecomposable RG-module, define a vertex D of M and a source M_0 of M. Prove that

(a) the vertices of M are G-conjugate;

(b) any two sources for M (with respect to the vertex D) are $N_G(D)$-conjugate;

(c) if the p'-part of $|G|$ is invertible in R, then the vertices of M are p-subgroups.

What does it mean to say that the RG-module M is a trivial source module? Show that the indecomposable module M has a trivial source if and only if it is a direct summand of a permutation module.

Use Mackey Decomposition to show that, if M_1 and M_2 are $\mathfrak{O}G$-permutation modules on the cosets of H_1 and H_2 respectively, then the natural homomorphism from $\mathrm{Hom}_{\mathfrak{O}G}(M_1, M_2)$ to $\mathrm{Hom}_{kG}(\bar{M}_1, \bar{M}_2)$ given by reduction modulo $\mathfrak{p}$ is surjective.

Deduce, using the Idempotent Refinement Theorem, that any trivial source kG-module lifts to a trivial source $\mathfrak{O}G$-module, unique up to isomorphism.

3 (a) Take $n \in \mathbb{N}$ and let $H = \mathbb{Z}/p^n$ be the cyclic group of order p^n, and k a field of characteristic p. Show that there are p^n isomorphism classes of indecomposable kH-modules $V_1, V_2, \ldots, V_{p^n}$ with $\dim(V_i) = i$, and that $\dim_k \mathrm{Hom}_{kH}(V_i, V_j) = \min\{i, j\}$.

(b) Let P be a direct product of two copies of the cyclic group of order p. If k is infinite, show that the group algebra kP has infinite representation type.

(c) Deduce, using the theory of vertices and sources, that a block of kG has a cyclic defect group if and only if there is only a finite number of indecomposable kG-modules lying in the block.

4 Suppose that G acts by conjugation on $\mathfrak{O}G$.

(a) Identify the fixed point space $(\mathfrak{O}G)^G$ and if $H \leqslant G$ identify an $\mathfrak{O}$-basis for $(\mathfrak{O}G)^H$. Define the transfer map Tr_H^G and state why the image $(\mathfrak{O}G)_H^G$ is an ideal in $Z(\mathfrak{O}G)$. Given a Sylow p-subgroup P of H, prove that $(\mathfrak{O}G)_H^G = (\mathfrak{O}G)_P^G$.

In what follows we work over k. Let D be an arbitrary p-subgroup of G.

(b) Prove that $(kG)^D = kC_G(D) \oplus \sum_{D'<D}(kG)_{D'}^D$ as a sum of a subring and a 2-sided ideal.

(c) Use this decomposition to define the Brauer homomorphism, Br_D. Write down the kernel of $\mathrm{Br}_D \downarrow_{(kG)^{N_G(D)}}$.

(d) Show that Br_D induces a one-to-one correspondence between block idempotents in $Z(kG)$ with defect group D and primitive idempotents in $(kC_G(D))_D^{N_G(D)}$, given by sending $e \in (kG)_D^G$ to $\mathrm{Br}_D(e)$ (results used should be clearly stated).

(e) Deduce Brauer's First Main Theorem, namely that if H is a subgroup of G containing $N_G(D)$, then there is a one-to-one correspondence between blocks of G with defect group D and blocks of H with defect group D.

5 (a) State carefully the Green Correspondence.

(b) Let e be a central idempotent in kG and M a kG-module; let D be a p-subgroup of G, and let K be a subgroup with $C_G(D) \leqslant K \leqslant N_G(D)$.

(i) With this notation, state and prove Nagao's version of Brauer's Second Main Theorem, under the assumption that $e.M = M$.

(ii) Suppose that M is indecomposable with vertex D and that e is primitive. Deduce that

$$e.M = M \iff \mathrm{Br}_D(e).M' = M'$$

where M' is the Green Correspondent of M as a $kN_G(D)$-module.

Let V be an indecomposable kK-module with vertex D such that $\mathrm{Br}_D(e).V = V$. Suppose that

$$V \uparrow^G = e.(V \uparrow^G) \oplus (\bigoplus_j V_j)$$

where V_j is indecomposable. Let D_j be a vertex of V_j. Prove that $D_j \leqslant_G D \cap {}^gD$ for some $g \in G \setminus N_G(D)$. (This last result is Juhász's version of Nagao's Theorem.)

6 In this question k is an infinite field. What does it mean to say that kG has finite representation type?

Suppose B is a block of kG whose defect group D is cyclic of order p^n. Let Q be the unique subgroup of D of order p. Define the inertial index of B. Write down the Green Correspondence between modules for G and $N_G(Q)$.

Assume that B has inertial index 1. Prove that there is only one simple module S in B, and that the projective cover of S is uniserial of length p^n. (You may assume the result at the end of question 3, and can also state any other general facts you need.)

END OF PAPER