UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 7 June 2002 1.30 to 4.30

PAPER 26

MODULAR FORMS

Attempt ***THREE*** *questions*

There are ***six*** *questions in total*

The questions carry equal weight

M_k *(resp.* S_k*) denotes the space of modular forms (resp. cusp forms) of weight* k *on* $SL_2(Z)$*.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For every $k \geqslant 2$, express the Eisenstein series

$$G_{2k}(\tau) = \sum_{m,n}{}' (m\tau + n)^{-2k} \qquad (\tau \in \mathcal{H})$$

and its normalized version $E_{2k}(\tau)$ as a power series in $q = e^{2\pi i \tau}$. Show that

$$G_{2k}(\tau) = \sum_{4a+6b=2k} c_{a,b}^{(k)} \, G_4^a \, G_6^b$$

for some $c_{a,b}^{(k)} \in \mathbf{Q}$. Express E_8, E_{10}, E_{14} in terms of E_4, E_6.

2 Let

$$D = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq} \qquad (\tau \in \mathcal{H},\ q = e^{2\pi i \tau}).$$

Show that

(i) If $f \in M_k$, $g \in M_\ell$, then $\ell g Df - kf Dg \in S_{k+\ell+2}$.

(ii) If $f \in M_k$, then $\left(D - \frac{k}{12} E_2\right) f \in M_{k+2}$ where $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, q^n$.

(iii) The coefficients $\tau(n)$ of

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n$$

satisfy

$$(1 - n)\tau(n) = 24 \sum_{k=1}^{n-1} \sigma_1(k) \tau(n-k)$$

and

$$\tau(n) \equiv n\, \sigma_5(n) \pmod 5.$$

3 Let $\chi : (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*$ be a primitive Dirichlet character with $N > 1$. Let

$$\theta_\chi(\tau) = \sum_{\substack{n\in\mathbf{Z}\\(n,N)=1}} \chi(n)\, e^{\pi i n^2 \tau}$$

and

$$\widetilde{\theta}_\chi(\tau) = \sum_{\substack{n\in\mathbf{Z}\\(n,N)=1}} \chi(n)\, n\, e^{\pi i n^2 \tau}.$$

State and prove a relation between $\theta_\chi(\tau)$ and $\theta_{\overline{\chi}}(-1/N^2\tau)$ when $\chi(-1) = 1$, and between $\widetilde{\theta}_\chi(\tau)$ and $\widetilde{\theta}_{\overline{\chi}}(-1/N^2\tau)$ when $\chi(-1) = -1$. Deduce that

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbf{Z}}(-1)^n q^{(3n+1)n/2}$$
$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n (2n+1)\, q^{n(n+1)/2}.$$

4 Write an account of the theory of Hecke operators acting on M_k.

5 Let $f(\tau) = \sum_{n=0}^{\infty} a_n\, q^n \in M_k$. State and prove the meromorphic continuation and functional equation for $(2\pi)^{-s}\Gamma(s)L(f,s)$, where

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

6 Let $z = x + iy \in \mathcal{H}$ and $\mathrm{Re}(s) > 1$. Develop the function

$$E(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s){\sum_{m,n}}' \frac{y^s}{|mz+n|^{2s}}$$

into Fourier expansion in terms of the function

$$K_s(Y) = \frac{1}{2}\int_0^\infty e^{-Y(t+t^{-1})/2}\, t^s\, \frac{dt}{t}.$$

Deduce: **either** the meromorphic continuation and functional equation of $E(z,s)$;

or Kronecker's limit formula.