UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 1.30 to 4.30

PAPER 75

MODULAR FORMS

Attempt at most ***THREE*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe the standard fundamental domain D for the action of $SL_2(\mathbf{Z})$ on the upper half plane $\mathcal{H}$ and show that it is, indeed, a fundamental domain. Determine all elliptic fixed points of $SL_2(\mathbf{Z})$ in $\overline{D}$ and their stabilizers.

2 Show that the ring of modular forms on $SL_2(\mathbf{Z})$ is generated by the holomorphic Eisenstein series of weights 4 and 6, which are algebraically independent over $\mathbf{C}$.

3 Define the Hecke operators $T(n)$ acting on $S_k(SL_2(\mathbf{Z}))$ and determine their action on Fourier expansions. If $f \in S_k(SL_2(\mathbf{Z}))$ satisfies $f|T(n) = \lambda(n)f$ for all $n \geqslant 1$, show that $L(f,s)$ admits an appropriate Euler product.

4 Let $f = \sum_{n\geqslant 1} a(n)q^n$ and $g = \sum_{n\geqslant 1} b(n)q^n$ be elements of $S_k(SL_2(\mathbf{Z}))$. Using the Rankin-Selberg method, express the function $F(s) = \sum_{n\geqslant 1} a(n)\overline{b(n)}n^{-s}$ in terms of f, g and the non-holomorphic Eisenstein series

$$E(z,s) = \pi^{-s}\Gamma(s) \sum_{\substack{c,d\in\mathbf{Z} \\ (c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}} \qquad (\mathrm{Re}(s) > 1).$$

Assuming the standard analytic properties of $E(z,s)$ (holomorphic continuation in s to $\mathbf{C} - \{0,1\}$; simple poles at $0,1$; invariance under $s \longleftrightarrow 1-s$), deduce similar properties of $F(s)$. If $L(f,s)$ and $L(g,s)$ admit an Euler product, show that the same holds for $F(s)$.

5 Applying the Riemann-Hurwitz formula to the projection $X(N) = \Gamma(N)\backslash\mathcal{H}^* \longrightarrow SL_2(\mathbf{Z})\backslash\mathcal{H}^*$, determine the genus of $X(N)$ $(N \geqslant 1)$.

6 Let $f = \sum_{n\geqslant 1} a(n)q^n \in S_k(\Gamma_0(N), \psi)$ and let χ be a primitive Dirichlet character modulo D, where $(N, D) = 1$. Show that $f_\chi = \sum_{n\geqslant 1} a(n)\chi(n)q^n$ lies in $S_k(\Gamma_0(D^2N), \chi^2\psi)$. Determine $f_\chi \Big| \begin{pmatrix} 0 & -1 \\ D^2N & 0 \end{pmatrix}$ in terms of $g = f \Big| \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, where

$$\left(f \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = (ad-bc)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$