UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 9 to 12

PAPER 47

MECHANICS OF ELASTIC SOLIDS

Attempt any ***FOUR*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that any real (3×3) matrix $\mathbf{A}$ with positive determinant can be written as

$$\mathbf{A} = \mathbf{RU} = \mathbf{VR} \ ,$$

where $\mathbf{R}$ is a rotation matrix and $\mathbf{U}$ and $\mathbf{V}$ are real symmetric matrices.

Given that the Cauchy stress $\boldsymbol{\sigma}$ in an isotropic body is a function of the deformation gradient $\mathbf{A}$, show that $\boldsymbol{\sigma}$ is coaxial with the left-stretch matrix. Show further that the principal stresses σ_1, σ_2 and σ_3 are symmetric functions of the principal stretches λ_1, λ_2 and λ_3.

2 A homogeneous, isotropic, elastic solid has a strain energy density $W(\mathbf{A})$, where $\mathbf{A}$ is the deformation gradient, such that the Piola-Kirchhoff stress tensor $\mathbf{s}$ is given by

$$s_{ij} = \rho_0 \frac{\partial W}{\partial A_{ji}} \ .$$

Show how the Lamé constants λ, μ for infinitesimal strain can be found from W.

Write down the equations of motion for plane (non-linear) waves propagating in the 1-direction of Cartesian co-ordinates. Establish the existence of transversely polarised waves (displacements in the 2-direction).

Construct the solution of the equations in terms of the characteristic curves and show how a complete solution can be constructed for a simple wave generated by tractions on the plane $x_1 = 0$. If the tractions increase smoothly from zero at $t = 0$, what is the condition for a shock to form?

3 Prove uniqueness of the solution of the equations of linear elastodynamics in a finite domain D with boundary S, with given body force and with initial displacement and velocity prescribed, and the following boundary conditions:

$$\sigma_{ij} n_j = k_{ij}(u_j - U_j) \ , \qquad \mathbf{x} \in S \ ,$$

where $\mathbf{U}(\mathbf{x}, t)$ is prescribed on S, $k_{ij} = k_{ji}$ and $k_{ij} u_i u_j \leq 0$ for all $\{u_i\}$; $\{\sigma_{ij}\}$ is the stress and $\mathbf{n}$ the outward normal to S.

Interpret physically the boundary conditions

$$\left.\begin{aligned} \sigma_{ij} n_j - n_i \sigma_{jk} n_j n_k &= \epsilon_{ijk} n_j F_k \ , \\ u_i n_i &= N \ , \end{aligned}\right\} \mathbf{x} \in S_1 \ ,$$

$$u_i = V_i \ , \qquad \mathbf{x} \in S_2 \ ,$$

where $\mathbf{F}(\mathbf{x}, t), N(\mathbf{x}, t)$ and $\mathbf{V}(\mathbf{x}, t)$ are prescribed on S, and $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$.

Prove uniqueness of the solution under these boundary conditions.

4 Show that the mean stress $\overline{\boldsymbol{\sigma}}$ in a material volume V with boundary S is given by

$$V\overline{\sigma}_{ij} = \int_S x_i t_j dS + \int_V x_i (b_j - a_j)\rho dV \ ,$$

where $\mathbf{t}$ is the traction on the boundary, $\mathbf{b}$ is the body force and $\mathbf{a}$ the acceleration.

Show also that the mean strain $\overline{\mathbf{e}}$ is given by

$$V\overline{e}_{ij} = \tfrac{1}{2}\int_S (u_i n_j + u_j n_i) dS \ .$$

Self-gravity is switched on in a homogeneous isotropic sphere of initial radius a, as a result of which the sphere deforms with infinitesimal strain. Show that the gravity field in the sphere corresponds to a body force

$$\mathbf{g} = -g\mathbf{x}/a \ ,$$

where g is the gravitational acceleration at the surface and $\mathbf{x}$ is the position vector relative to the centre of the sphere.

Write down the relation between the mean stress and mean strain in the sphere, given that the bulk modulus of the material is κ and the shear modulus μ. Show that the decrease in radius of the sphere is

$$\rho g a^2/15\kappa \ .$$

5 The time-harmonic Green function $G_i^j(\mathbf{x},\boldsymbol{\xi})e^{i\omega t}$ for an unbounded and homogeneous medium is given by

$$G_i^j(\mathbf{x},\boldsymbol{\xi}) = \frac{\eta_i\eta_j}{4\pi\alpha^2 R}\, e^{-i\omega R/\alpha} \quad + \quad \frac{(\delta_{ij}-\eta_i\eta_j)}{4\pi\beta^2 R}\, e^{-i\omega R/\beta}$$
$$- \frac{i}{4\pi\omega R^2}\left[\frac{e}{\alpha}^{-i\omega R/\alpha}\left(1-\frac{i\alpha}{\omega R}\right) - \frac{e}{\beta}^{\,i\omega R/\beta}\left(1-\frac{i\beta}{\omega R}\right)\right](3\eta_i\eta_j - \delta_{ij}) \ ,$$

where α, β are the wave speeds of compressional and shear waves respectively, and

$$R = |\mathbf{x}-\boldsymbol{\xi}| \,, \quad \boldsymbol{\eta} = (\mathbf{x}-\boldsymbol{\xi})/R \ .$$

Under what circumstances can the third term in the expression for G_i^j be neglected?

Describe and sketch the radiation pattern (amplitude variation with direction of the receiver) in the far field for compressional and shear waves from a time-harmonic point force in the direction $\mathbf{d}$, acting at the origin.

A point slip is represented by a double couple; that is, a body force $\mathbf{b}$ of the form

$$b_1 = \frac{\partial}{\partial\xi_2}\left\{\delta(\mathbf{x}-\boldsymbol{\xi})\right\} e^{i\omega t}$$
$$b_2 = \frac{\partial}{\partial\xi_1}\left\{\delta(\mathbf{x}-\boldsymbol{\xi})\right\} e^{i\omega t}$$
$$b_3 = 0 \ .$$

Find the corresponding radiation patterns in the far field.

An explosion source is given by

$$b_i = \frac{\partial}{\partial\xi_i}\left\{\delta(\mathbf{x}-\boldsymbol{\xi})\right\} e^{i\omega t} \ .$$

What are the radiation patterns this time? Why?

6 Starting from the equation of motion

$$c_{ijpq}\,\frac{\partial^2 u_p}{\partial x_q \partial x_j} \;=\; \rho\,\frac{\partial^2 u_i}{\partial t^2}$$

for the displacement in a uniform anisotropic material with elastic constants $\{c_{ijpq}\}$ and density ρ, derive the equation for the wave speeds of plane harmonic waves. Show that all the wave speeds are real. Describe the slowness surface.

Show that, if the displacements are written as

$$\mathbf{u} = \mathbf{a}\exp(i\omega s_3 x_3)\exp i\omega\left\{(n_1x_1 + n_2x_2)/c - t\right\}\ ,$$

where $\mathbf{a}, c, \mathbf{n} = (n_1, n_2, 0)$ are real constants with $n_1^2 + n_2^2 = 1$, there are between zero and six real roots of s_3, depending on the values of n_1/c and n_2/c.

Taking the $z-$axis as the vertical, show how to construct waves of the most general form

$$\mathbf{u} = \mathbf{U}(x_3)\exp i\omega\left\{(n_1x_1 + n_2x_2)/c - t\right\}\ ,$$

which are either (a) upgoing, or (b) downgoing, where c, n_1 and n_2 are real.

Derive expressions for the wave field due to a plane harmonic wave incident on the plane free surface of a homogeneous, anisotropic, elastic half-space. Show how the wave amplitudes may be calculated.

7 A point source at the surface of a spherically symmetric Earth in which the wave speeds vary smoothly, radiates along ray paths along which the quantity p, given by $p = r\sin i/v$, is a constant where r is the distance to the centre of the Earth, i is the angle between the tangent to the ray and the vertical and v is the wave speed. The travel-time of a wave which is recorded at the Earth's surface at epicentral distance $\triangle$ is $T(\triangle)$. Show that

$$\frac{dT}{d\triangle} = p\ .$$

Derive an expression for $\triangle$ as an integral along a ray. Show (by establishing the Herglotz-Wiechert method) that the wave speed can be found as a function of r from the $\triangle - p$ relation for surface earthquakes so long as $\eta = r/v$ is a strictly increasing function of r.

Does this method work if η decreases with r in some range? If not, explain why.

[You may use the fact that

$$\int_{\mu}^{\eta}\frac{p\,dp}{(p^2-\mu^2)^{1/2}\,(\eta^2-p^2)^{1/2}} \;=\; \frac{\pi}{2}\,,\ \text{if}\ \ \mu < \eta\ .\Big]$$