UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 13:30 to 16:30

PAPER 33

Mathematics of Operational Research

Attempt **FOUR questions.**

There are **six questions in total.**

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the linear program

$$P: \quad \text{maximize} c^{\top} x\,, \quad Ax \leqslant b \quad \text{and} \quad x \geqslant 0.$$

Here $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is $m \times n$. Derive the dual problem, D.

Prove or provide a counterexample to each of the following.

(a) If P is unbounded then D is infeasible.

(b) If P is infeasible then D is unbounded.

The following is the final simplex algorithm tableau for a linear programming problem P, in which $n = m = 2$. Find all the optimal solutions to both the primal and the dual problems.

What was the original primal problem?

x_1	x_2	z_1	z_2	
0	1	$\frac{1}{10}$	$\frac{1}{10}$	1
1	0	$\frac{1}{20}$	$\frac{3}{20}$	2
0	0	0	$-\frac{1}{2}$	-3

2 Given a graph (N, A) with flow capacities on the arcs, and nodes $s, s' \in N$ it is desired to maximize the flow from s to s'. Assuming the theorem that min-cut equals max-flow, prove that the Ford-Fulkerson algorithm solves this problem.

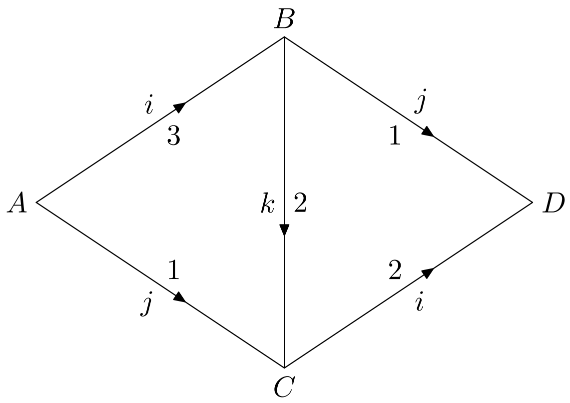
A total of n teams play in a football league. Thus far in the season team i has won w_i games. Teams i and j are still to play one another in g_{ij} games. We want to know if it is possible for team n to end the season having won more games than any other team. Explain how this problem can be addressed via a maximum flow problem in which, for each $i \neq j$, $i, j < n$, a node s is connected to a node r_{ij} with an arc of capacity g_{ij}, each node r_{ij} is also connected to nodes t_i and t_j with arcs of infinite capacity, and each node t_i is connected to a node s' with an arc of capacity $w^* - w_i - 1$ and $w^* = w_n + \sum_{i<n} g_{in}$.

Suppose that, regardless of the size of n, every parameter of the problem can be represented in no more than k bits. Is the problem in $\mathcal{P}$?

3 Write an essay on the minimum spanning tree problem. Carefully explain what is meant by complexity classes $\mathcal{P}$ and $\mathcal{NP}$ and prove that the minimum spanning tree problem belongs to both of these.

4 Define the terms *characteristic function* and *core* of a cooperative game.

In the network below the five directed links between pairs of nodes A, B, C, D are owned by three communications companies i, j and k, as marked. The maximum number of units of data flow that can be carried on each link is shown by an integer beside the link. Customers will pay £6000 per unit of data flow from A to D (irrespective of the path it takes) and £4000 per unit of data flow from B to C. By formulating an appropriate minimum cost network flow problem, show that the maximum possible revenue is £22000.



The companies are trying to reach an agreement about how much data traffic should be carried and how the resulting revenue should be shared. Specify by a set of constraints all the ways that they could share the revenue of £22000 such that no subset of companies would have any incentive to prefer operating without the others.

Viewing the above as a cooperative game, find the nucleolus.

5 40 jobs are to be processed sequentially on a single machine in some order. The processing time of job i is t_i. If job i is the first job to be processed on the machine then a time s_i will be required to set up the machine. If job j immediately follows job i then a time s_{ij} will be required to change tools on the machine. Show how the problem of finding the schedule that minimizes the time to complete all the jobs can be formulated as a 0–1 linear programming problem.

Describe how you could address this problem by

(a) a branch and bound algorithm, and the solution of a sequence of assignment problems;

(b) a simulated annealing algorithm, using a 2-opt heuristic.

Suppose now that three machines are available to work on the jobs in parallel. Suggest a strategy for tackling this problem.

6 Consider a n-person game in which each player has a finite number of pure strategies. Define what is meant by a *Nash equilibrium*.

Show that a Nash equilibrium in pure strategies may not exist.

50000 students are applying to 50 universities. The students can be ranked $1, \ldots, 50000$ and each student knows his own rank. Each student is permitted to apply to exactly one university. Once all applications are in, each university accepts up to 1000 students, choosing those with the greatest ranks amongst all those who apply. An accepted student obtains a benefit equal to the average rank of those students who are accepted at his university. A student who is not accepted obtains benefit 0. Show that there are exactly 50! different Nash equilibria in pure strategies.

Suppose now that we allow mixed strategies. Does this lead to 0, 1, or more than 1, further Nash equilibrium?