UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 1.30 to 4.30

PAPER 34

MATHEMATICS OF OPERATIONAL RESEARCH

Attempt ***FOUR*** *questions*

There are ***six*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the ILP

$$\begin{aligned} &\text{minimize} \quad 3x_1 + 4x_2 \\ &\text{subject to} \\ &\qquad 3x_1 + x_2 \geqslant 4 \\ &\qquad x_1 + 2x_2 \geqslant 4 \\ &\qquad x_1, x_2 \geqslant 0 \\ &\qquad x_1, x_2 \text{ integer} \end{aligned}$$

Ignoring the integer constraints, the following tableau gives the optimal solution.

x_1	x_2	z_1	z_2	
1	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{4}{5}$
0	1	$\frac{1}{5}$	$-\frac{3}{5}$	$\frac{8}{5}$
0	0	$-\frac{2}{5}$	$-\frac{9}{5}$	$\frac{44}{5}$

Explain Gomory's cutting plane method, illustrating it by showing that from the above tableau one can deduce that the optimal integer solution must satisfy the additional constraint

$$\tfrac{1}{5}z_1 + \tfrac{2}{5}z_2 \geqslant \tfrac{3}{5}.$$

Use this constraint and the dual simplex algorithm to find the optimal solution to the ILP.

2 Let A be a $m \times n$ matrix of integers and let b be a vector in $\mathbb{R}^m$. Let U be the largest absolute value of the entries of A and b. By using Cramer's rule or otherwise, prove that every extreme point of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$ satisfies

$$-n!U^n \leqslant x_j \leqslant n!U^n, \quad j = 1, \ldots, n.$$

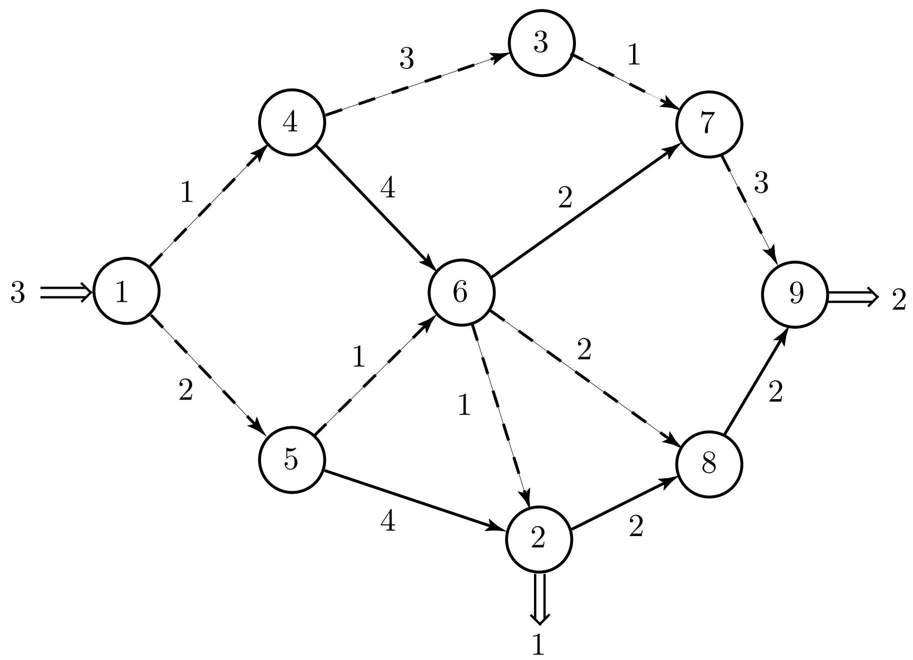
Give an account of the ellipsoidal algorithm for the problem of deciding whether or not P is empty. Describe the inputs to the algorithm and its main steps. You need not derive any detailed formulae, but you should explain enough so that the role of the above result is clear.

3 Define the uncapacitated minimum cost network flow problem.

Define the Lagrangian for this problem. Derive dual feasibility and complementary slackness conditions that can be used to identify an optimal flow. Decribe how a spanning tree can be used to calculate a feasible choice of dual variables.

In the network below, the label next to each arc is the cost per unit flow on that arc, c_{ij}. A flow of b_i enters at node i, where $b_1 = 3$, $b_2 = -1$, $b_9 = -2$ and $b_i = 0$, $i \neq 1, 2, 9$. Find the arc flows, f_{ij}, for the basic solution corresponding to the spanning tree indicated by the dashed lines.

Starting from this basic solution, explain the network simplex algorithm and show that it finds the optimal flow in one step.



4 Define the terms characteristic function, imputation, core and Shapley value payoffs as they apply to n-person coalitional games.

Consider a market in which there four participants. Player 1 values his car at 0 and wishes to sell it. Each of Players 2, 3 and 4 wishes to buy the car, and each values it at 1. Find the characteristic function, core and Shapley value payoffs.

Consider now a market in which there are $4k$ participants, of which each of k participants has a car to sell and each of the other $3k$ participants is a potential buyer of one car. All cars are identical, of value 0 to sellers and value 1 to buyers. Show that as $k \to \infty$ the Shapley value payoff of a seller tends to 1 and the Shapley value payoff of a buyer tends to 0.

5 An instance of the ΔTSP *decision problem* is an undirected graph (with all possible edges present), a nonnegative integer cost $c_{ij} = c_{ji}$ for each edge $\{i, j\}$ and a nonnegative integer L. Edge costs are required to satisfy the triangle inequality. The question is whether there is a tour whose cost is no greater than L. Show that this problem is in the complexity class $\mathcal{NP}$.

The ΔTSP *evaluation problem* is defined on the same instances (but omitting L), and the problem is to find the length of the shortest tour. Show that there exists a polynomial time algorithm for this problem if and only if there exists a polynomial time algorithm for the decision problem.

An instance of HCP is a graph G (with only some of the possible edges present). The question is whether there exists a tour that visits each vertex exactly once (a Hamiltonian circuit). Show that if HCP is $\mathcal{NP}$-complete then the ΔTSP decision problem is also $\mathcal{NP}$-complete.

Given the same data as a ΔTSP evaluation problem, the ΔTSP *optimization problem* is to find a minimum length tour; the MST optimization problem is to find a minimum spanning tree. Show that if there exists a polynomial time algorithm for the MST optimization problem then there also exists a polynomial time 1-approximation algorithm for the ΔTSP optimization problem, such that it produces a tour no more than twice the length of the minimal length tour.

6 Define what is meant by an equilibrium pair for a non-zero-sum two-person game.

State conditions under which at least one equilibrium pair is guaranteed to exist.

Two friends have different preferences for composers. Without consulting one another, they must each book for one of three possible concerts. They are pleased if they happen to book for the same concert. This is modelled by a game with the following payoff matrix.

	Bach	Mozart	Schubert
Bach	$(4, 2)$	$(1, 1)$	$(0, 0)$
Mozart	$(0, 0)$	$(2, 4)$	$(0, 0)$
Schubert	$(0, 0)$	$(0, 0)$	$(3, 3)$

Find all the equilibrium pairs.