UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 7 June, 2004 9 to 11

PAPER 35

MATHEMATICAL MODELS IN FINANCIAL MANAGEMENT

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) With the aid of a diagram briefly explain what the terms *opportunity set*, *efficient frontier*, *market portfolio*, *capital market line* and *market price of risk* (or *Sharpe ratio*) mean in the context of Markowitz portfolio theory. You may assume here and in the sequel that short sales are permitted.

Consider an economy with a risk-free asset S_0 and three risky assets S_1, S_2 and S_3. The risk-free return on S_0 is $r = 3$ percent and the expected returns (in percent) on the risky assets are

$$R_1 = 10, \quad R_2 = 17, \quad R_3 = 24$$

and the covariance matrix of returns for the risky assets is

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 9 & 2 \\ 1 & 2 & 16 \end{pmatrix}.$$

(b) Show that the market portfolio and the market price of risk can be determined by solving the linear system

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 9 & 2 \\ 1 & 2 & 16 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ 21 \end{pmatrix}.$$

(c) Deduce that the solution of this linear system is

$$\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 13 \\ 13 \\ 12 \end{pmatrix}.$$

(d) Hence, show that the market portfolio consists of proportions of total wealth $w_i,\ i = 1, 2, 3$ invested in risky asset $S_i,\ i = 1, 2, 3$, respectively, where

$$w_1 = \tfrac{13}{38}, \quad w_2 = \tfrac{13}{38}, \quad w_3 = \tfrac{12}{38},$$

(e) Show that expected return R_{M} and risk σ_{M} associated with the market portfolio are respectively

$$R_{\mathrm{M}} = \tfrac{639}{38}, \quad \sigma_{\mathrm{M}} = \tfrac{5}{38}\sqrt{231}.$$

(f) Hence deduce that the market price of risk in this economy is

$$\theta = \tfrac{105}{\sqrt{231}}.$$

(g) Suppose that the risk-free rate was changed to $r = 25$, but all other parameters remained the same. Briefly explain how the result of the computations performed with this new set of parameters would differ qualitatively from the computations with the old set of parameters.

2 (a) An investor has a utility function $U(W)$, where W is their wealth. They may play a game in which their wealth changes to $W(1+\boldsymbol{\phi})$ where $\boldsymbol{\phi}$ is a small random variable with zero mean and variance $\sigma_\phi^2 << 1$, or they may sacrifice a proportion ψ of their wealth to avoid playing the game, in which case their wealth changes to $W(1-\psi)$. Deduce that, approximately, ψ is related to σ_ϕ^2 by

$$\psi = -\frac{1}{2}\frac{WU''(W)}{U'(W)}\sigma_\phi^2.$$

(b) What does it mean to say that the investor has a constant level of relative risk aversion? For a given constant level α of relative risk aversion, find the utility function and determine whether the investor is risk-averse, risk-neutral or risk-seeking for that level.

At time t an investor has a portfolio of value Π_t consisting of bonds to the value B_t and a risky asset. You may assume that the value of the portfolio evolves according to

$$d\boldsymbol{\Pi}_t = rB_t dt + \mu(\boldsymbol{\Pi}_t - B_t)dt + \sigma(\boldsymbol{\Pi}_t - B_t)d\mathbf{X}_t,$$

where r is the constant risk-free rate, $\mu > r$ is the constant drift of the risky asset, σ is the risky asset's constant volatility and $d\mathbf{X}_t$ is the increment of a standard Brownian motion (Wiener process) during the infinitesimal time increment dt. The investor wishes to invest over a finite time horizon, $0 \leqslant t \leqslant T$, and can control the investment strategy by varying B_t over time. Their aim is to maximize the following combined utility function:

$$\max_{B_t} \mathbb{E}_0 \left[\int_0^T e^{-\rho t}\boldsymbol{\Pi}_t^\gamma dt + \boldsymbol{\Pi}_T^\gamma \right],$$

where $0 < \gamma < 1$ and $\rho > 0$ are constants and $\mathbb{E}_t$ denotes the expected value with the information available at time t.

(c) By setting

$$J(\Pi_t, t) := \max_{B_t} \mathbb{E}_t \left[\int_t^T e^{-\rho\tau}\boldsymbol{\Pi}_\tau^\gamma d\tau + \boldsymbol{\Pi}_T^\gamma \right]$$

and applying Bellman's principle, deduce that the optimal investment strategy is given by

$$B_t^\star = \Pi_t + \frac{\mu - r}{\sigma^2}\frac{\partial J/\partial\Pi}{\partial^2 J/\partial\Pi^2},$$

where J satisfies the final value problem

$$e^{-\rho t}\Pi^\gamma + \frac{\partial J}{\partial t} + r\Pi\frac{\partial J}{\partial \Pi} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(\partial J/\partial\Pi)^2}{\partial^2 J/\partial\Pi^2} = 0, \quad J(\Pi, T) = \Pi^\gamma.$$

(d) A solution of this problem of the form $J(\Pi, t) = \Theta(t)X(\Pi)$ gives

$$B_t^\star = \left(1 + \frac{(\mu - r)}{(\gamma - 1)\sigma^2}\right)\Pi_t.$$

Explain in financial terms the behaviour of $B_t^\star$ as a function of γ.

3 (a) Define what it means for a discrete time stochastic process $\{\mathbf{M}_n : 0 \leqslant n \leqslant N\}$ with filtration $\mathcal{F}_n$ to be a $\mathbb{P}$-martingale.

(b) Show that $\mathbb{E}[\mathbf{M}_n] = \mathbb{E}[\mathbf{M}_0]$ for all n.

(c) Consider a financial market consisting of two assets: $\mathbf{S}$ is the price of a risky asset and the other is a bank account of size B which grows at the risk-free rate.

A *previsible* trading strategy $(\boldsymbol{\phi}_n, \boldsymbol{\psi}_n) \in \mathcal{F}_{n-1} \times \mathcal{F}_{n-1}$ is said to be *self-financing* if the value $\mathbf{V}_n$ of the portfolio $(\boldsymbol{\phi}_n, \boldsymbol{\psi}_n)$ at time n satisfies

$$\mathbf{V}_n = \boldsymbol{\phi}_n \mathbf{S}_n + \boldsymbol{\psi}_n B_n = \boldsymbol{\phi}_{n+1} \mathbf{S}_n + \boldsymbol{\psi}_{n+1} B_n.$$

(i) Explain what is meant by previsible and interpret the self-financing condition.

(ii) Show that the self-financing condition is equivalent to

$$\mathbf{V}_n - \mathbf{V}_{n-1} = \boldsymbol{\phi}_n (\mathbf{S}_n - \mathbf{S}_{n-1}) + \boldsymbol{\psi}_n (B_n - B_{n-1}).$$

(iii) Show that if the trading strategy $(\boldsymbol{\phi}_n, \boldsymbol{\psi}_n)$ is self-financing, then it is self-financing in the discounted asset price $B_n^{-1}\mathbf{S}_n$.

(d) Now suppose that there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ which makes $B_n^{-1}\mathbf{S}_n$ a martingale. Assume that $\mathbf{X}$ is the payoff of a European option with maturity N written on the asset price $\mathbf{S}$. Show that $B_n^{-1}\mathbf{V}_n$ is also a martingale and hence derive the option pricing formula

$$V_0 = B_N^{-1} \mathbb{E}_{\mathbb{Q}}[\mathbf{X}].$$

4 (a) If the cumulative distribution function F has a known inverse denoted F^{-1}, then for a sequence, $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$, uniformly distributed in $[0, 1]$ show that elements of the sequence $\{\boldsymbol{\eta}_i\} := \{F^{-1}(\boldsymbol{\xi}_i)\}$ are distributed F.

(b) Given a set of uniformly distributed pseudo-random numbers, how would one derive a set of normally distributed random variables.

(c) Use the standard risk neutral valuation of a European call option with strike price X under the Black Scholes assumptions to show that its value is given by

$$C(S,t) = e^{-r(T-t)} \int_{\mathrm{lnX}}^{\infty} (e^u - X) f_{\ln \mathbf{S}_T}(u) du.$$

Show that this expression can be converted to involve an integral of the form

$$\int_0^1 g(\xi) d\xi$$

and explain how this integral can be approximated using Monte Carlo techniques.

(d) Briefly explain three variance reduction techniques and why they help to improve the accuracy of Monte Carlo estimates.

5 (a) Consider a pure discount bond maturing at time T with current price $P(t,T)$. Define the bond's *yield to maturity* $R(t,T)$, the *instantaneous forward rate* $f(t,T)$ and the *spot rate* $r(t)$ and set out the relations between them. Explain the notion of the *yield curve* given by $R(t,\cdot)$.

(b) Suppose the dynamics of the spot rate are given by

$$d\mathbf{r}_t = b(t,\mathbf{r}_t)dt + \sigma(t,\mathbf{r}_t)d\mathbf{W}_t,$$

where $\mathbf{W}_t$ is the Wiener process. Let $V(r,t)$ denote the current *price of a derivative security* with terminal payoff $\mathbf{X}$ at T and no intermediate payments. Suppose $\mathbf{V}$ has dynamics of the form

$$d\mathbf{V}_t = m(t,\mathbf{r}_t)dt + s(t,\mathbf{r}_t)d\mathbf{W}_t,$$

with drift and volatility determined by applying Ito's lemma to $V(t,\mathbf{r}_t)$ under suitable technical conditions. Show that the *market price of short rate risk* defined by

$$\theta(t,r) := [m(t,r) - rV(t,r)]/s(t,r)$$

is independent of the derivative security price process $\mathbf{V}$.

(c) Use this result to derive the Cox-Ross PDE for the interest rate derivative security price $V(t,r)$ equivalent to the Black-Scholes PDE for an equity derivative price as

$$\frac{1}{2}\sigma^2(t,r)\frac{\partial^2 V(t,r)}{\partial r^2} + [b(t,r) - \sigma(t,r)\theta(t,r)]\frac{\partial V(t,r)}{\partial r} + \frac{\partial V(t,r)}{\partial t} - rV(t,r) = 0.$$

(d) Under the *risk neutral measure* Q the spot rate dynamics are given by

$$d\mathbf{r}_t = (\mathbf{b}_t - \boldsymbol{\theta}_t\boldsymbol{\sigma}_t)dt + \boldsymbol{\sigma}_t d\widetilde{\mathbf{W}}_t$$

and the derivative price by

$$V(t,r) = \mathbb{E}_Q\left[e^{-\int_t^T \mathbf{r}_s ds}\mathbf{X}|\mathbf{r}_t = r\right].$$

Find an expression for $P(t,T)$ and use a martingale argument and Ito's lemma to give an alternative derivation of the Cox-Ross PDE when $\mathbf{r}$ is Markov.