UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Monday 5 June, 2006 9 to 12

PAPER 69

MAGNETOHYDRODYNAMICS AND TURBULENCE

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Randomly tangled magnetic fields in galaxy clusters can be measured because the polarisation angle of an electromagnetic wave propagating from an extended radio source in a cluster rotates as it passes though magnetised intracluster medium. Let z be the line-of-sight direction and (x, y) the coordinates in the plane perpendicular to it. The rotation angle is $\Delta\phi = \lambda^2\sigma(x, y)$, where λ is the wavelength and

$$\sigma(x, y) = a_0 \int_{z_{\rm source}}^{z_{\rm observer}} dz\, n_e B_z$$

is called the *rotation measure*. Here a_0 is a constant, n_e is the electron density in the intracluster medium and B_z is the projection of the magnetic field in the medium onto the line of sight. While B_z is a (random) function of x, y, and z, you may assume that n_e is constant. Let us assume that we have a two-dimensional data set with values of $\sigma(x, y)$ for all x and y. Your task is to determine the spectrum of magnetic energy based on this information and some statistical assumptions.

(a) Assuming spatial homogeneity of the field, the two-point correlation function of the magnetic field depends only on the distance between points: $\langle B_i(\mathbf{r})B_j(\mathbf{r}')\rangle = C_{ij}(\mathbf{r} - \mathbf{r}')$. Define the Fourier transform

$$\hat{\mathbf{B}}(\mathbf{k}) = \int d^3r\, \mathbf{B}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{r} = (x, y, z).$$

Using the formula $\int d^3r\, e^{-i\mathbf{r}\cdot\mathbf{k}} = (2\pi)^3\delta(\mathbf{k})$, show that $\langle \hat{B}_i(\mathbf{k})\hat{B}_j(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\hat{C}_{ij}(\mathbf{k})$, where

$$\hat{C}_{ij}(\mathbf{k}) = \int d^3r\, C_{ij}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}.$$

(b) Now assume spatial isotropy and mirror symmetry of the field. Then the tensor $\hat{C}_{ij}(\mathbf{k})$ can be written in terms of one scalar function of $k = |\mathbf{k}|$. If $\hat{C}_{ii}(\mathbf{k}) = 2H(k)$, what is the expression for $\hat{C}_{ij}(\mathbf{k})$ in terms of $H(k)$ and $\mathbf{k}$?

(c) Suppose we have constructed from our data set the correlation function of the rotation measure, i.e., $C_{\rm RM} = \langle\sigma(\mathbf{r}_{\perp 1})\sigma(\mathbf{r}_{\perp 2})\rangle$ is known for any two points $\mathbf{r}_{\perp 1} = (x_1, y_1)$ and $\mathbf{r}_{\perp 2} = (x_2, y_2)$. Show that $C_{\rm RM}$ and $H(k)$ are related as follows

$$C_{\rm RM}(|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|) = a_0^2 n_e^2 L \int \frac{d^2k_\perp}{(2\pi)^2} H(|\mathbf{k}_\perp|)\, e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_{\perp 1}-\mathbf{r}_{\perp 2})}, \quad \mathbf{k}_\perp = (k_x, k_y),$$

where L is the distance from the source to the observer. You are allowed to take the integration limits in z to $\pm\infty$ wherever you need to. You may use the formula $\int_{-\infty}^{+\infty} dz\, e^{ik_z z} = 2\pi\delta(k_z)$.

(d) Now show that the spectrum of the magnetic field can be recovered from the observed rotation-measure correlation function as follows:

$$H(k) = \frac{2\pi}{a_0^2 n_e^2 L}\int_0^\infty dr r J_0(kr) C_{\rm RM}(r).$$

You may use the formula $\int_0^{2\pi} d\theta\, e^{\pm ia\cos\theta} = 2\pi J_0(a)$.

2 Interstellar and solar-wind turbulence at scales smaller than $d_i = c(m_i/4\pi e^2 n)^{1/2}$ (the ion skin depth; here m_i is ion mass, e elementary charge, n number density of ions/electrons) can be described by an approximation whereby the magnetic field is frozen into the electron flow $\mathbf{u}_e$, while the ions can be considered motionless, $\mathbf{u}_i = 0$. Since the current density is $(c/4\pi)\nabla \times \mathbf{B} = \mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e) = -en\mathbf{u}_e$, this turbulence obeys the following equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi e n}\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}]\,. \qquad (1)$$

(a) Consider a static equilibrium with a straight uniform magnetic field in the z direction, so that $\mathbf{B} = B_0\hat{\mathbf{z}} + \delta\mathbf{B}$. Show that the linear waves in this system have the dispersion relation

$$\omega(\mathbf{k}) = \pm k_\parallel v_A k d_i,$$

where $v_A = B_0/\sqrt{4\pi n m_i}$, $\mathbf{k} = (k_\perp, 0, k_\parallel)$, $k = |\mathbf{k}|$. These waves are called Kinetic Alfvén Waves (KAW).

(b) Now your task is to work out scalings for the KAW turbulence in a way similar to how this is done for the Alfvén-wave turbulence. As usual, assume that interactions in scale space are local and the energy flux is constant:

$$\epsilon \sim \left(\frac{\delta B_l}{B_0}\right)^2 \frac{v_A^2}{\tau_l} \sim \text{const},$$

where τ_l is the cascade time, which you will have to determine. In terms of the typical magnetic-field fluctuation δB_l of scale l, what is the characteristic time associated with the nonlinearity in Eq. (1)? Formulate the assumption of weak interactions. Under this assumption, calculate τ_l and show that

$$\frac{\delta B_l}{B_0} \sim \left(\frac{\epsilon}{v_A^3 d_i}\frac{l_\perp^3}{l_\parallel}\right)^{1/4},$$

where $l_\perp$ and $l_\parallel$ are characteristic scales perpendicular and parallel to the background field, respectively.

(c) If the turbulence were isotropic, what would be scaling of the spectrum of the magnetic field with the wavenumber?

(d) Now let us assume that the turbulence is anisotropic, $l_\parallel \gg l_\perp$, and critically balanced, i.e., the interactions are strong and the time for a wave to cascade is comparable to the wave period. Show that in this case, $\delta B_l \propto l_\perp^{2/3}$ and $l_\parallel \propto l_\perp^{1/3}$. This is the picture confirmed by numerical simulations.

3 If Ohmic diffusion is ignored the magnitude of a randomly advected magnetic field, $\tilde{B}(t) = |\mathbf{B}(t)|$, satisfies the following equation

$$\frac{d\tilde{B}}{dt} = \tilde{\sigma}\tilde{B}, \qquad (1)$$

where $\tilde{\sigma} = \hat{\mathbf{b}} \cdot (\nabla \mathbf{u}) \cdot \hat{\mathbf{b}}$ ($\mathbf{u}$ is the velocity field, $\hat{\mathbf{b}}$ is the magnetic-field direction). We shall model the quantity $\tilde{\sigma}(t)$ by the following simple equation

$$\frac{d\tilde{\sigma}}{dt} = -\frac{1}{\tau}\,\tilde{\sigma} + f(t), \qquad (2)$$

where τ is the velocity decorrelation time and $f(t)$ is a Gaussian white noise whose correlation function satisfies $\langle f(t)f(t')\rangle = \kappa\delta(t-t')$. Note that there is no explicit dependence on the space variable anywhere in our model.

(a) First show that $\tilde{\sigma}(t)$ is a Gaussian random process with correlation time τ: write explicitly the time-dependent solution of Eq. (2) [you may assume $\sigma = 0$ at $t = 0$] and calculate the correlation function $\langle\tilde{\sigma}(t)\tilde{\sigma}(t')\rangle$, where $t \geq t' \gg \tau$. You should find that it decays exponentially with $t - t'$ and its characteristic width is τ.

(b) Let $\tilde{P} = \delta(B - \tilde{B}(t))\delta(\sigma - \tilde{\sigma}(t))$. Define the joint PDF of $\tilde{B}$ and $\tilde{\sigma}$: $P(B, \sigma, t) = \langle\tilde{P}\rangle$. Derive an evolution equation for this PDF using the Furutsu-Novikov formula

$$\langle f(t)\tilde{P}(t)\rangle = \int dt'\, \langle f(t)f(t')\rangle \left\langle \frac{\delta\tilde{P}(t)}{\delta f(t')} \right\rangle.$$

(c) It will turn out that the moments of the magnetic-field strength grow exponentially:

$$\langle B^n\rangle(t) = \int_{-\infty}^{+\infty} d\sigma \int_0^{\infty} dB\, B^n P(B, \sigma, t) \propto e^{\gamma_n t}.$$

Your task is to calculate γ_n. However, you cannot obtain a closed evolution equation for $\langle B^n\rangle(t)$ simply by multiplying by B^n both sides of the evolution equation you have derived for P and integrating with respect to B and σ. Why not?

It is convenient to define the following auxiliary quantity:

$$P_n(\sigma, t) = \int_0^{\infty} dB\, B^n P(B, \sigma),$$

so that $\langle B^n\rangle(t) = \int_{-\infty}^{+\infty} d\sigma P_n(\sigma, t)$. Show that P_n satisfies the following equation

$$\frac{\partial P_n}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 P_n}{\partial \sigma^2} + \frac{1}{\tau}\frac{\partial}{\partial \sigma}(\sigma P_n) + n\sigma P_n.$$

(d) Now seek solutions of this equation in the form $P_n(\sigma, t) = e^{\gamma_n t} p_n(\sigma)$. This gives an eigenvalue problem for $p_n(\sigma)$. Convert this eigenvalue problem into the standard eigenvalue problem for a harmonic oscillator and find γ_n. The answer is $\gamma_n = \kappa\tau^2 n^2/2$.

Hint. Use the ansatz $p_n(\sigma) = \psi(\sigma) \exp\left(-\sigma^2/2\kappa\tau\right)$. The equation for p_n should then reduce to the Schrödinger equation for a harmonic oscillator:

$$\frac{d^2\psi}{dx^2} + (2E - x^2)\psi = 0,$$

where x is the appropriately defined new variable (related to σ) and E is related to γ_n, κ, τ and n. Since P_n cannot take negative values, the energy must be set to its ground state value $E = 1/2$. This should give you the expression for γ_n.

4 Kolmogorov-Richardson theory of turbulence.

(a) Describe the line of reasoning that leads to the Kolmogorov scaling of the velocity increments $\delta u_l \sim l^{1/3}$ for hydrodynamic turbulence. Be sure to state all assumptions that you consider essential. What is the scale range in which the Kolmogorov spectrum holds?

(b) Consider two fluid particles initially separated by an infinitesimal distance l_0. Derive and sketch the time dependence of the spatial separation between these particles in a field of fluid turbulence: how does the particle separation $l(t)$ depend on time when (i) l is smaller than the viscous scale, (ii) l larger than the viscous scale but smaller than the outer scale, (iii) l larger than the outer scale? How long does stage (ii) last?

END OF PAPER