UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 9 to 12

PAPER 43

MAGNETIC FIELDS IN STARS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

Candidates may bring their own notebooks into the examination

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain the distinction between lack of equilibrium and instabilities associated with magnetic buoyancy.

A plane stratified atmosphere contains a perfect gas with density $\rho(z)$, pressure $p(z)$ and temperature $T(z)$, and a magnetic field $\mathbf{B} = B_0(z)\hat{\mathbf{y}}$, referred to cartesian axes with the z-axis pointing vertically upward. Consider a bodily displacement of a horizontal flux tube in the xz-plane. Given that heat transfer is so efficient that the displaced tube always remains in thermal equilibrium with its surroundings, find the condition for the atmosphere to be unstable to interchanges driven by magnetic buoyancy.

2 A turbulent conducting fluid occupies an infinite conducting slab of depth d in the presence of a sheared horizontal velocity $\mathbf{u} = (\pi V z/d)\hat{\mathbf{y}}$, referred to cartesian co-ordinates with the z-axis vertical. The mean magnetic field does not vary in the y-direction and can be decomposed into a poloidal component in the xz-plane, described by a flux function $A(x, z, t)$, and a toroidal component $B(x, z, t)\hat{\mathbf{y}}$. The equations describing a kinematic $\alpha\omega$-dynamo are

$$\frac{\partial A}{\partial t} = \alpha B + \eta \nabla^2 A, \qquad \frac{\partial B}{\partial t} = \left(\frac{\pi V}{d}\right)\frac{\partial A}{\partial x} + \eta \nabla^2 B.$$

What is the significance of the various terms in these equations?

Find the minimum value of the dynamo number $D = \frac{\alpha V d^2}{\pi^2 \eta^2}$ for which there exists an exponentially growing dynamo wave solution of the form

$$A = \tilde{A}(t)\exp[i(kx + \pi z/d)], \quad B = \tilde{B}(t)\exp[i(kx + \pi z/d)].$$

What is the corresponding critical value of the wave number k?

3 The evolution of the velocity $\mathbf{u}$, the temperature T and the magnetic field $\mathbf{B}$ in Boussinesq magnetoconvection is governed by the equations

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho_r}\boldsymbol{\nabla}p + \frac{\rho}{\rho_r}\mathbf{g} + \frac{1}{\rho_r}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + \nu\nabla^2\mathbf{u},\\
&\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})T = \kappa\nabla^2 T,\\
&\frac{\partial \mathbf{B}}{\partial t} - \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) = \eta\nabla^2\mathbf{B},\\
&\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \rho = \rho_r[1-\alpha(T-T_r)].
\end{aligned}$$

Consider two-dimensional motion in the region $\{0 < x < \lambda d; 0 < z < d\}$ in the presence of an imposed vertical field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, referred to cartesian co-ordinates with the z-axis pointing upwards, subject to the boundary conditions

$$\begin{aligned}
&u_z = \partial u_x/\partial z = B_x = 0 \qquad \text{on } z = 0, d\\
&u_x = \partial u_z/\partial x = B_x = \partial T/\partial x = 0 \quad \text{on} \quad x = 0, \lambda d
\end{aligned}$$

and $T(x,0) = T_r$, $T(x,d) = T_r - \Delta T$. Show by adopting a suitable scaling that the stream function $\psi(x,z,t)$, the temperature perturbation $\Theta(x,z,t)$ and the perturbation to the flux function $\chi(x,z,t)$ are related by

$$\begin{aligned}
&\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} = \sigma\left[\nabla^4\psi + R\frac{\partial\Theta}{\partial x} + \zeta Q\frac{\partial(A,\nabla^2 A)}{\partial(x,z)}\right],\\
&\frac{\partial\Theta}{\partial t} + \frac{\partial(\psi,\Theta)}{\partial(x,z)} = \frac{\partial\psi}{\partial x} + \nabla^2\Theta,\\
&\frac{\partial\chi}{\partial t} + \frac{\partial(\psi,\chi)}{\partial(x,z)} = \frac{\partial\psi}{\partial z} + \zeta\nabla^2\chi,
\end{aligned}$$

where R, Q, σ, ζ are the Rayleigh number, the Chandrasekhar number, the Prandtl number and the ratio of magnetic to thermal diffusivity, respectively.

Now set $p = \pi^2(1+\lambda^2)/\lambda^2$, $\quad \tau = pt$, $\quad r = \pi^2 R/\lambda^2 p^3$, $\quad q = \pi^2 Q/p^2$ and consider linearized perturbations of the form

$$\begin{aligned}
\psi(x,z,\tau) &= \frac{p\lambda}{\pi}\sin\frac{\pi x}{\lambda}\sin\pi z\, a(\tau),\\
\Theta(x,z,\tau) &= \cos\frac{\pi x}{\lambda}\sin\pi z\, b(\tau),\\
\chi(x,z,\tau) &= \lambda\sin\frac{\pi x}{\lambda}\cos\pi z\, c(\tau),
\end{aligned}$$

and show that

$$\dot{a} = \sigma[-a + rb - \zeta qc], \quad \dot{b} = -b + a, \quad \dot{c} = -\zeta c + a.$$

Hence find conditions for the occurrence of stationary (pitchfork) and oscillatory (Hopf) bifurcations from the trivial static solution with $\psi = \Theta = \chi = 0$.

4 Discuss the processes that lead to concentration and expulsion of magnetic flux by eddies in a convecting layer. How do these processes relate to magnetic fields at the surface of the Sun?