UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 9 to 12

PAPER 56

LOCAL AND GLOBAL BIFURCATIONS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The normal form for a particular codimension-2 bifurcation is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\lambda x + \kappa y - x^3 - x^2 y, \end{aligned} \tag{1}$$

where λ and κ are real parameters.

(a) Locate the equilibrium points of (1) and sketch the curves in the (κ, λ) plane along which local bifurcations occur, classifying each bifurcation.

(b) Use the rescaling $x = \varepsilon u$, $y = \varepsilon^2 v$, $\lambda = \varepsilon^2 \alpha$, $\kappa = \varepsilon^2 \beta$, $\tau = \varepsilon t$ to deduce ODEs for u and v which, in the limit $\varepsilon \to 0$, have the conserved quantity $H(u, v) = \frac{1}{2}v^2 + \frac{\alpha}{2}u^2 + \frac{1}{4}u^4$. Sketch contours of constant H in the (u, v) plane when $\alpha < 0$. Give the value of H which corresponds to the homoclinic orbits.

(c) By integrating around one of the homoclinic orbits for small ε, find the relation between α and β, and hence between λ and κ, at the global bifurcation. Indicate this curve on your sketch of the (κ, λ) plane from part (a).

(d) Compute the saddle index $\delta = \frac{-m_-}{m_+}$ near the global bifurcation, where $m_+ > 0 > m_-$ are the eigenvalues of the linearisation at the relevant saddle point. Briefly describe the dynamics near the global bifurcation in this case.

(e) Using the result of part (d) and the fact that exactly one of the Hopf bifurcations is supercritical and the other is subcritical, sketch the phase portrait of (1) in the six regions of the (κ, λ) plane which display qualitatively distinct behaviour.

2 Consider the following system of ODEs:

$$\begin{aligned} \dot{u} &= \mu u + u^2 - uv, \\ \dot{v} &= \lambda(u^2 - v), \end{aligned} \tag{1}$$

where μ and λ are real parameters and $\lambda > 0$.

(a) Determine the location of all the equilibria of (1) and conditions for their existence.

(b) Describe the steady-state bifurcations that occur at $\mu = -\frac{1}{4}$ and at $\mu = 0$ for fixed λ. Sketch the appropriate bifurcation diagrams in the (μ, u) plane, indicating stabilities in each case.

(c) Identify the equilibrium which undergoes a Hopf bifurcation (which you may assume is supercritical) and find the location of the curve $\mu = \mu_H(\lambda)$ along which this occurs. Sketch this, and the locations of the steady-state bifurcations identified in part (b), in the (μ, λ) plane. Find the codimension-2 bifurcation point.

(d) Near the codimension-2 point the dynamics are described by the normal form for a Takens-Bogdanov bifurcation without symmetry:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 + \mu_2 y + x^2 - xy. \end{aligned}$$

Recall that the analysis of this bifurcation involves the location of exactly one curve of global bifurcations. Indicate a possible location (near the codimension-2 point) for this curve of global bifurcations on your sketch from part (c) - you do not need to carry out any calculations.

(e) Sketch the dynamics of (1) in the part of the (u, v) plane where $u > 0$ for the four regions of qualitatively distinct behaviour around the codimension-2 point.

(f) Explain why the periodic orbit that is created at the Hopf bifurcation cannot be destroyed at the global bifurcation when $\mu > \mu_H$ and $\lambda > 1$, and sketch the behaviour of trajectories in the part of the (u, v) plane where $u > 0$ for these parameter values.

(g) The periodic orbit is found to exist throughout the region $\mu > 0$ and $\frac{1}{4} < \lambda < 1$ but does not exist when $\lambda < \frac{1}{4}$. Its disappearance on the line $\lambda = \frac{1}{4}$, $\mu > 0$ is closely related to the behaviour of $W^u(0,0)$ on this line. Near $W^u(0,0)$ the system (1) can be well-approximated by the system

$$\begin{aligned} \dot{u} &= \mu u + u^2 - uv, \\ \dot{v} &= \lambda u^2. \end{aligned} \tag{2}$$

First introduce the nonlinear time rescaling

$$\frac{d}{dt} \mapsto u \frac{d}{d\tau}$$

which linearises (2). Integrate the resulting (inhomogeneous) linear system, using the initial condition $(u, v) = (h, v_0)$ where both h and v_0 are small and positive. Show that the travel time (in the new variable τ) around the trajectory before u becomes small again is approximately $\frac{2\pi}{\sqrt{4\lambda - 1}}$ when $\lambda > \frac{1}{4}$. Hence interpret the behaviour of $W^u(0,0)$ and the periodic orbit, as λ decreases through $\frac{1}{4}$.

3 Write an essay on complex dynamics created near global bifurcations in $\mathbb{R}^3$. You should discuss both the 'Lorenz' scenario (in the symmetric case, where the dynamics is invariant under the symmetry $(x, y, z) \to (-x, -y, z)$) and the 'Shilnikov' scenario, using the notes below.

(a) In the symmetric 'Lorenz' case take the linearisation around the saddle-point $(0, 0, 0)$ to be

$$\dot{x} = \lambda_+ x,$$
$$\dot{y} = \hat{\lambda} y,$$
$$\dot{z} = \lambda_- z,$$

where $\lambda_+ > 0 > \lambda_- \gg \hat{\lambda}$, and let the two branches of the one-dimensional unstable manifold $W^u(0, 0, 0)$ intersect the plane Σ given by $z = h$ at the points $(-\mu, \nu, h)$ and $(\mu, -\nu, h)$ respectively. Derive a return map $\Pi_L : \Sigma \to \Sigma$ and show that it may be approximated by the one-dimensional map

$$x_{n+1} = f_L(x_n) \equiv \text{sgn}(x_n)(-\mu + A|x_n|^\delta) \tag{1}$$

where $\delta = \frac{-\lambda_-}{\lambda_+}$, A is a constant and $\text{sgn}(x) = x/|x|$. You may take A to be positive. Discuss the dynamics of the map (1) in the two cases $\delta > 1$ and $\delta < 1$.

(b) In the 'Shilnikov' case, take the linearisation around $(0, 0, 0)$ to be

$$\dot{x} = \lambda_- x - \omega y,$$
$$\dot{y} = \omega x + \lambda_- y,$$
$$\dot{z} = \lambda_+ z,$$

where $\omega > 0$ and $\lambda_+ > 0 > \lambda_-$. Let the unstable manifold $W^u(0, 0, 0)$ intersect the plane Σ given by $y = 0$ at the point $(r, \theta, z) = (\rho, 0, -\mu)$ in cylindrical polar co-ordinates. Derive a return map $\Pi_S : \Sigma \to \Sigma$ and show that it may be approximated by the one-dimensional map

$$x_{n+1} = f_S(x_n) \equiv -\mu + A x_n^\delta \cos\left(\frac{\omega}{\lambda_+} \log(x_n) + \Phi\right), \tag{2}$$

where $\delta = \frac{-\lambda_-}{\lambda_+}$ and A and Φ are positive constants. Discuss the dynamics of the map (2) in the two cases $\delta > 1$ and $\frac{1}{2} < \delta < 1$.

4 Two-dimensional thermal convection in a porous medium, in the presence of a vertical magnetic field, is governed by the equations

$$\frac{\partial}{\partial t}\nabla^4\psi + \frac{\partial(\psi,\nabla^4\psi)}{\partial(x,z)} = \sigma\nabla^6\psi - \sigma R\frac{\partial^2\psi}{\partial x^2} + \sigma\zeta Q\frac{\partial}{\partial z}\nabla^4 A,$$
$$\frac{\partial A}{\partial t} + \frac{\partial(\psi,A)}{\partial(x,z)} = \zeta\nabla^2 A + \frac{\partial\psi}{\partial z},$$

for the streamfunction $\psi(x,z,t)$ and the magnetic flux function $A(x,z,t)$. The Prandtl number σ, magnetic Prandtl number ζ, Rayleigh number R and Chandrasekhar number Q are all positive constants. R is proportional to the imposed temperature difference across the fluid layer and Q is proportional to the square of the imposed magnetic field. The fluid is confined to the region $0 \leqslant x \leqslant \sqrt{3}$ and $0 \leqslant z \leqslant 1$. The boundary conditions imposed are $\psi = \nabla^2\psi = 0$ on all four walls, $\frac{\partial A}{\partial z} = 0$ on $z = 0, 1$ and $A = 0$ on $x = 0, \sqrt{3}$.

(a) Use the three Fourier modes $\sin\alpha x\sin\pi z$, $\sin\alpha x\cos\pi z$, and $\sin 2\alpha x$ to describe the expected spatial structure of convection cells near onset, where $\alpha = \pi/\sqrt{3}$, and hence derive the ODEs

$$\begin{aligned} \dot{a} &= -\sigma a + \sigma r a - \sigma\zeta q b, \\ \dot{b} &= a - \zeta b - ac, \\ \dot{c} &= -\zeta c + 3ab, \end{aligned} \tag{1}$$

for the corresponding mode amplitudes $a(t)$, $b(t)$ and $c(t)$, which approximate the dynamics near the onset of convection. In (1) the notation $q = 9Q/16\pi^2$ and $r = 9R/64\pi^4$ has been used and the new timescale $\tau = \frac{4\pi^2}{3}t$ has been introduced.

(b) Show that local bifurcations occur in the ODEs (1) at $r = 1 + q$ and $r = 1 + \zeta/\sigma$. Locate the codimension-2 point.

(c) Using either adiabatic elimination or a centre manifold reduction, reduce the ODEs (1) to a single equation

$$\dot{a} = C_1\mu a + C_2 a^3, \tag{2}$$

that describes the bifurcation at $r = 1 + q$ when $q < \zeta/\sigma$. C_1 and C_2 are constants to be determined, and the parameter μ is proportional to $r - 1 - q$. Classify this bifurcation fully and sketch the relevant bifurcation diagram in the (μ, a) plane.

(d) Explain why, from a physical point of view, (2) gives an unsatisfactory description of the dynamics when $r > 1 + q$. When three-dimensional effects are included an appropriate ODE for the amplitude of convection is now

$$\dot{a} = C_1\mu a - a^2 + C_2 a^3 - a^5. \tag{3}$$

Briefly discuss the dynamics of (3), highlighting qualitative differences between its dynamics and those of (2). You may find it helpful to sketch a bifurcation diagram for (3).