UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2001 1.30 to 3.30

PAPER 51

LOCAL AND GLOBAL BIFURCATIONS

*Attempt **TWO** questions. The questions are of equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Use the near-identity transformation

$$u = x + \alpha_1 x^3 + \beta_1 x^2 y + \gamma_1 x y^2 + \delta_1 y^3$$
$$v = y + \alpha_2 x^3 + \beta_2 x^2 y + \gamma_2 x y^2 + \delta_2 y^3$$

to reduce the following second-order system of ODEs at a Takens–Bogdanov point

$$\dot{u} = v + a_1 u^3 + b_1 u^2 v + c_1 u v^2 + d_1 v^3$$
$$\dot{v} = a_2 u^3 + b_2 u^2 v + c_2 u v^2 + d_2 v^3$$

to the normal form (truncated at cubic order)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= P x^3 + Q x^2 y \end{aligned} \tag{1}$$

and determine the constants P and Q as functions of the eight original constants $a_1, \ldots, d_2$. Give your choices of the undetermined coefficients $\alpha_1, \ldots, \delta_2$ explicitly.

(b) The FitzHugh–Nagumo equations describe the nonlinear propagation of nerve impulses along axons. A simplification of the FitzHugh–Nagumo model leads to the following system of three ordinary differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - 2y + z + \frac{1}{3}x^3 \\ \dot{z} &= -\frac{c}{2}x + \frac{d}{2}z \end{aligned} \tag{2}$$

where biological constraints imply that the parameters c and d are positive. Identify the codimension-2 point (c^*, d^*).

(c) Sketch, in the (c, d) plane, the curves along which codimension-1 bifurcations from the trivial solution take place.

(d) Investigate the local bifurcations from any non-trivial equilibria of (2) you find, and add this information to your sketch.

(e) In order to investigate the dynamics near (c^*, d^*) in more detail, set $c = c^*$ and $d = d^*$ and introduce the new variables $v = z - x$ and $w = (2v - y)/3$ into the third-order system (2), changing coordinates from (x, y, z) to (z, v, w).

Explain why v is a fast variable and so may be eliminated via either centre manifold reduction or adiabatic elimination. Why would neither procedure yield any quadratic terms in the expression for the centre manifold $v = h(z, w)$?

(f) After a centre manifold reduction and a near-identity transformation of the type described in part (a) have been carried out, the second-order system can be put into the form (1).

Using your answer to part (a), carry out this centre manifold reduction to eliminate v including only those terms needed to subsequently compute P and Q. Hence calculate the values of P and Q for this case.

2 Consider two-dimensional Boussinesq convection in the presence of a vertical magnetic field, in the region $0 \leq x \leq L$ and $0 \leq z \leq 1$. The dimensionless governing equations for the streamfunction $\psi(x,z,t)$, the perturbation $\theta(x,z,t)$ to the conductive temperature profile and the magnetic flux function $A(x,z,t)$ are:

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,z)} = \sigma\nabla^4\psi + \sigma r R_0 \frac{\partial \theta}{\partial x} + \sigma\zeta Q\left(\frac{\partial}{\partial z}\,(\nabla^2 A) + \frac{\partial(A, \nabla^2 A)}{\partial(x,z)}\right)$$
$$\frac{\partial \theta}{\partial t} + \frac{\partial(\psi, \theta)}{\partial(x,z)} = \nabla^2\theta + \frac{\partial \psi}{\partial x}$$
$$\frac{\partial A}{\partial t} + \frac{\partial(\psi, A)}{\partial(x,z)} = \zeta\nabla^2 A + \frac{\partial \psi}{\partial z}$$

where Q is proportional to the square of the imposed magnetic field, σ is the Prandtl number of the fluid, ζ is the magnetic Prandtl number and $R_0 = \beta^6/\alpha^2$ with $\alpha = \pi/L$ and $\beta^2 = \alpha^2 + \pi^2$. The boundary conditions are chosen for convenience: $\psi = 0$ on all boundaries, $\theta = \partial A/\partial z = 0$ on $z = 0, 1$ and $\partial\theta/\partial x = A = 0$ on $x = 0, L$.

(a) Adopt the truncated representation

$$\psi = 2\sqrt{2}\frac{\beta}{\alpha}a(t)\sin\alpha x\sin\pi z$$
$$\theta = 2\sqrt{2}\frac{1}{\beta}b(t)\cos\alpha x\sin\pi z - \frac{1}{\pi}c(t)\sin 2\pi z$$
$$A = \frac{2\sqrt{2}\pi}{\alpha\beta}d(t)\sin\alpha x\cos\pi z + \frac{1}{\alpha}e(t)\sin 2\alpha x$$

and derive the following ODEs that describe the weakly nonlinear dynamics:

$$\dot{a} = -\sigma a + \sigma r b - \sigma\zeta q d - \sigma\zeta q(3 - \varpi)ed$$
$$\dot{b} = a - b - ac$$
$$\dot{c} = \varpi(-c + ab)$$
$$\dot{d} = a - \zeta d - ae$$
$$\dot{e} = -(4 - \varpi)\zeta e + \varpi a d$$

where $\varpi = 4\pi^2/\beta^2$ and $q = \pi^2 Q/\beta^4$. The dots denote differentiation with respect to the scaled time $\tilde{t} = \beta^2 t$. Note that σ, ζ and ϖ are positive constants and $\varpi < 4$.

(b) Locate and classify the steady-state bifurcation from the trivial solution and determine the conditions under which a Hopf bifurcation can occur. Find the Takens–Bogdanov point in the (r, q) plane.

(c) Find r as a function of a^2 on the branch of non-zero steady solutions. Assuming $a^2 \ll \zeta^2$, approximate your expression in the form $r = 1 + q + a^2 r_2 + \mathcal{O}(a^4)$. Use the resulting expression for r_2 to explain what happens to the branch of steady solutions as ϖ passes through values close to 2 when $\zeta^2 \ll 1$.

3 A second-order system of ODEs near a codimension-2 bifurcation is described by the following normal form:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\lambda + \mu y + x^2 + xy \end{aligned} \tag{1}$$

where λ and μ are parameters.

(a) Find the equilibria and classify their local bifurcations. Sketch the lines of local bifurcations in the (λ, μ) plane and sketch the phase portraits of the system (1) in the regions of this plane which the local bifurcations indicate have qualitatively different behaviour. You may assume that the Hopf bifurcation is subcritical.

(b) Adopt the scaling $x = \varepsilon^2 u$, $y = \varepsilon^3 v$, $\lambda = \varepsilon^4 \alpha$, $\mu = \varepsilon^2 \beta$ and rescale time by a factor of $1/\varepsilon$. Find the resulting ODEs for u and v. In the limit $\varepsilon \to 0$, these ODEs have a conserved quantity $H(u, v) = \frac{1}{2}v^2 + \alpha u - \frac{1}{3}u^3$. Show that $\mathrm{d}H/\mathrm{d}t = 0$, sketch contours of constant H in the (u, v) plane and find the value of H which corresponds to a homoclinic orbit in the system.

(c) By integrating around the homoclinic orbit for small ε find the relation between α and β and hence between λ and μ at the global bifurcation, when ε is small. Show this line of global bifurcations in your sketch of the (λ, μ) plane and, assuming that there are no further bifurcations in the system (1), indicate the phase portraits on either side of this line.

(d) Denote the eigenvalues at the saddle point by $m_- < 0 < m_+$. Compute the saddle index $\delta = -m_-/m_+$ near the global bifurcation and verify that it is less than 1. Explain how this agrees with the analysis earlier in the question.