UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2001 9 to 12

PAPER 1

LIE GROUPS

*Candidates should attempt **TWO** questions from Section A and **ONE** question from Section B.*

Section A and Section B carry a maximum of 50 and 40 marks respectively.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 Let G be a Lie group and LG its associated Lie algebra. Define the exponential map

$$\exp : LG \to G.$$

Show (a) that exp is a local diffeomorphism at the origin $\mathbf{O}$ in LG, and (b) that exp is a homomorphism if and only if G is abelian. Deduce that if G is connected and abelian, then $G \cong \mathbb{R}^a \times T^b$, where T^b is a torus.

2 (a) Let $L(SL_2(\mathbb{R}))$ denote the Lie algebra of the group $SL_2(\mathbb{R})$. By considering the values taken by the trace of $\exp(A)$ ((For $A \in L(SL_2(\mathbb{R}))$) or otherwise, show that for this non-compact group the exponential map is not surjective.

(b) Prove the following sequence of propositions

(i) If m is a natural number and $A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 0 \neq t \in \mathbb{R}$, then

$$\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} A(t) \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}^{-1} = A(m^2 t) = A(t)^{m^2}.$$

(ii) Let $\phi : SL_2(\mathbb{R}) \to U_n$ be a finite dimensional unitary representation. Then the eigenvalues of the matrix $\phi(A(t))$ are obtained by permuting their m^2 powers for any m, and hence must all equal 1.

(iii) The normal subgroup generated by the matrices $\{A(t)) : t \in \mathbb{R}\}$ equals $SL_2(\mathbb{R})$.

(iv) The unitary representation ϕ in (ii) must be trivial.

3

Let G be a compact Lie group. Define the complex representation ring $R(G)$ and the ring of class functions $c\ell(G)$. If $[M] \to \mathcal{X}_M$ is the map which associates to each isomorphism class of G-modules M the character of M, give a careful proof that

$$\mathcal{X} : R(G) \to c\ell(G)$$

is a monomorphism. Use the group SU_2 to illustrate your answer.

Section B

4 Prove the following form of the Peter-Weyl theorem: every continuous function $f : G \to \mathbb{C}$, where G is a compact Lie group, can be approximated by functions of the form

$$\text{Trace}\ (\alpha\theta(g)),$$

where $\theta : G \to \text{Aut}_{\mathbb{C}}(M)$ is a homomorphism and $\alpha \in \text{Hom}_{\mathbb{C}}(M, M)$, for some finite-dimensional $\mathbb{C}$-vector space M.

Indicate briefly how this result implies that G admits a faithful representation in some unitary group U_n.

[*Any theorems from analysis to which you appeal, need not be proved, but they should be clearly stated.*]

5

Define what is meant by a maximal torus T in a compact, connected Lie group G. Outline the main steps in the proof that if g is an arbitrary element of G, then g is contained in some subgroup conjugate to T. Show further that the homomorphism

$$\text{Res}_{G \to T} : R(G) \to R(T)$$

is a monomorphism, whose image is contained in the subset of elements of $R(T)$ which are invariant under the action of the Weyl group.

What does this result say in the special case $G = SO(2m+1)$? Justify your answer.