UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 9 to 11

## PAPER 48

## LARGE-SCALE ATMOSPHERE–OCEAN DYNAMICS

*A distinction mark can be gained for complete and well-reasoned answers to **TWO** out of the three questions. The questions are of equal weight.*
*Clarity and explicitness of reasoning will attract more credit than perfection of computational detail.*

*$(x, y, z)$ denotes right-handed Cartesian coordinates and $(u, v, w)$ the corresponding velocity components; $t$ is time; the gravitational acceleration is $(0, 0, -g)$ where $g$ is a positive constant.*
*The fluid is always incompressible, and always ideal in the sense that viscosity can be neglected, and likewise buoyancy diffusion if relevant. $N$ denotes the buoyancy frequency of a stratified fluid.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Define 'strictly geostrophic flow'. Prove the Taylor–Proudman theorem for such flow, concerning $\partial(p, \mathbf{u})/\partial z$ where $z$ is the coordinate in the direction parallel to $\mathbf{\Omega} = (0, 0, \Omega)$, the angular velocity of the frame of reference, and where $p$ and $\mathbf{u}$ are appropriate pressure and relative velocity fields.

State briefly (without justification) how the property of $\partial(p, \mathbf{u})/\partial z$ asserted by the Taylor–Proudman theorem must be modified when the flow is confined between rigid boundaries that are nearly but not quite perpendicular to $\mathbf{\Omega}$ and when the flow goes across geostrophic contours. State briefly (again without justification) how the $z$-component of vorticity must then behave, indicating how that behaviour is described by the vorticity equation.

Assume now that the two rigid boundaries are exactly perpendicular to $\mathbf{\Omega}$, and spaced apart by a constant distance $h$, except in a central circular area $x^2 + y^2 < a^2$ in which the spacing is reduced by a small constant amount $\epsilon h$. Fluid flows through the central area, the flow at large $r^2 = x^2 + y^2$ being uniform and unidirectional, with velocity $(U, 0, 0)$. Assuming that the $z$-component of vorticity behaves as before, find the relative vorticity field and show that the relative velocity field has streamfunction

$$\psi \;\propto\; \begin{cases} -\frac{1}{2}r^2 & (r < a), \\ a^2(\log a - \frac{1}{2} - \log r) & (r > a). \end{cases}$$

Find the constant of proportionality in terms of $\epsilon$. Deduce the relative velocity field and find the value of $\epsilon/U$ such that closed streamlines are on the point of appearing in the relative velocity field.

**2** Linearize the shallow-water momentum and mass-conservation equations about a state of relative rest and uniform depth $h_{00}$, in a frame of reference rotating with angular velocity $(0, 0, \frac{1}{2}f)$, where $f$ is a positive constant. Deduce that the horizontal divergence $\delta = \nabla_H \cdot \mathbf{u} = \partial u/\partial x + \partial v/\partial y$ satisfies

$$\frac{\partial \delta}{\partial t} = fq - g\nabla_H^2 \zeta$$

where $\zeta$ is the surface elevation and $q$ the relative vorticity. Deduce also that

$$\frac{\partial q}{\partial t} = -f\delta$$

and that

$$\frac{\partial}{\partial t}\left(\frac{q}{h_{00}} - \frac{f\zeta}{h_{00}^2}\right) = 0 \ .$$

Comment briefly on the significance of the quantity $(f + q)/(h_{00} + \zeta)$ for the *nonlinear* shallow-water equations, and on how is it related to the last equation above.

Show that small disturbances with the plane-wave structure $\exp(ikx + ily - i\omega t)$ satisfy the linearized equations provided that either $\omega = 0$ or

$$\omega^2 = f^2 + c_0^2(k^2 + l^2) \ ,$$

where $c_0$ is a constant to be defined. Sketch $\omega$ against $(k^2 + l^2)^{1/2}$ for this last relation and comment briefly on the way in which the phase and group velocities vary when $\omega$ takes values near and far from $f$. In the case of disturbances for which $\omega = 0$, how are the $q$ and $\zeta$ fields related? In what qualitative way does the $\delta$ field differ from the $q$ and $\zeta$ fields, when $\omega = 0$?

Briefly discuss what happens, according to the linearized equations, when the system is started from relative rest with surface elevation $\zeta = \epsilon h_{00} \exp(-x^2/L^2)$, where $L$ is a given constant and $\epsilon \ll 1$.

**3** Write an essay on quasi-geostrophic flow in a stratified, rotating system. Possible points are

- brief motivation;
- scaling assumptions and small parameters;
- scale-analytic arguments: 'thermal-wind relations' and representation of the velocity and buoyancy fields in terms of a geostrophic streamfunction $\psi$;
- simplification of the vorticity equation to describe the evolution of the motion with time;
- description of the evolution in terms of potential-vorticity advection and inversion;
- modification of Taylor-Proudman effects by stratification; Prandtl's ratio of scales;
- the possibility of, and implications of, a horizontal gradient of the Coriolis parameter $f$.