UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS    Part III

Monday 13 June, 2005    9 to 11

# PAPER 36

# LARGE DEVIATIONS AND QUEUES

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

*While rigorous answers are preferred, heuristic answers will still gain partial credit.*

*You may find helpful the reference material at the end of the paper.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $A$ be an exponential random variable with mean $1/\lambda$. Let $B_N$ be the sum of $N$ independent normally distributed random variables with mean $\mu$ and variance $\sigma^2$. Let $\varepsilon > 0$.

(a) Find a large deviations principle for $A/N^{1+\varepsilon}$.

(b) Find a large deviations principle for $B_N/N^{1+\varepsilon}$.

(c) Find a large deviations principle for $(A + B_N)/N^{1+\varepsilon}$.

The rate functions you find should be non-trivial. (A rate function $I$ is said to be non-trivial if there is some $\hat{x}$ such that $0 < I(\hat{x}) < \infty$.) *Hint. What are the appropriate speeds for these LDPs?*

**2** Let $(T_1, T_2, \ldots)$ be a sequence of independent exponential random variables, where $T_i$ has mean $1/\lambda i$.

(a) Find a large deviations principle for $(T_1 + \cdots + T_k)/N$, where $k$ is fixed.

(b) Find a large deviations principle for $(T_1 + \cdots + T_k + Z_N)/N$, where $Z_N$ is the sum of $N - k$ independent copies of $T_k$ and $k$ is fixed.

(c) Hence (or otherwise) find a large deviations principle for $(T_1 + \cdots + T_N)/N$.

**3** Internet traffic is known to exhibit long-range correlations, and a popular model for this is *fractional Brownian motion*. Say that $Z(t), t \geqslant 0$, is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if it satisfies these properties:

(i) $Z$ is a Gaussian process, and $Z/\sqrt{L}$ satisfies an LDP with speed $L$ and some good rate function $I$ in the space $(\mathcal{C}_0, \|\cdot\|)$;

(ii) $Z$ is *self-similar*, i.e. for any $a > 0$, $a^{-H} Z^{\circlearrowright a}$ has the same distribution as $Z$, where $Z^{\circlearrowright a}(t) = Z(at)$.

Consider now a queue with constant service rate $C$ and infinite buffer, fed by the arrival process $X(-t, 0] = \mu t + \sigma Z(t)$, where $\mu < C$. The queue size is $r(X) = \sup_{t \in \mathbb{R}_+} X(-t, 0] - Ct$.

(a) Find an LDP for $Z^{\circlearrowright N}/N$.

(b) Find an LDP for $r(X)/N$.

(c) Let $J$ be the rate function from part (b). Show that $J(q) = q^{2(1-H)} J(1)$. *Hint. Write down an LDP for $r(X)/aN$, for $a > 0$.*

State clearly any general results you appeal to.

**4** BT wants to build an integrated packet network which carries both voice and data. Voice traffic needs low delays and low loss rates; data traffic can tolerate larger delays and larger loss rates. To accomodate these requirements, BT plans to build its network out of packet switches which behave as follows. There is a single queue. When this queue is full, all incoming packets are lost. When the queue is not full, all incoming data packets are admitted, and incoming voice packets are admitted only if their queueing delay will be smaller than some threshold value. Priority is given to serving voice packets. In order to ensure that both types of traffic get good quality of service, BT plans to limit the number of flows of each type.

(a) Explain what is meant by *effective bandwidth*. In your answer, state any relevant theorems. *Note. You may use without proof the fact that the effective bandwidth function is increasing.*

(b) Use the theory of effective bandwidth to model this system. Given the service rate, buffer size, maximum voice queueing delay, and maximum loss rates for the two types of traffic, how can BT calculate how many flows of each type may be carried? Explain your modelling assumptions.

**Reference: Gärtner-Ellis theorem**

A convex function $\Lambda : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is *essentially smooth* if

(i) the interior of its effective domain is non-empty

(ii) $\Lambda(\cdot)$ is differentiable throughout the interior of its effective domain

(iii) $\Lambda(\cdot)$ is steep, namely, $|\nabla\Lambda(\theta_n)| \to \infty$ whenever $(\theta_n)$ is a sequence in the interior of the effective domain converging to a point on the boundary of the effective domain.

Let $(X_L,\ L \in \mathbb{N})$ be a sequence of random vectors in $\mathbb{R}^d$, let $\alpha > 0$, and let

$$\Lambda^L(\theta) = \frac{1}{L^\alpha} \log \exp(L^\alpha \theta \cdot X_L)$$

for $\theta \in \mathbb{R}^d$. Assume that for each $\theta$ the limit

$$\Lambda(\theta) = \lim_{L\to\infty} \Lambda^L(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$. Assume further that 0 is in the interior of the effective domain of $\Lambda$, and that $\Lambda$ is essentially smooth and lower-semicontinuous. Then $(X_L,\ L \in \mathbb{N})$ satisfies an LDP in $\mathbb{R}^d$ at speed $L^\alpha$ with good rate function

$$\Lambda^*(x) = \sup_{\theta\in\mathbb{R}^d} \theta \cdot x - \Lambda(\theta).$$

## END OF PAPER