UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 9.00 to 12.00

## PAPER 87

## LAMBDA–CALCULUS

*Attempt* ***THREE*** *questions.*

*There are* ***FIVE*** *questions in total.*

*The questions carry equal weight.*

*For all $m < n$, results which you have proved in answering question $m$ (or which would have been proved if you had attempted question $m$) may be assumed in your answer to question $n$.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Explain what is meant by a $\lambda$-term, and by $\beta$-equivalence of two $\lambda$-terms. Prove the Church–Rosser theorem, that two $\lambda$-terms are $\beta$-equivalent if and only if they have a common $\beta$-reduct. Deduce that an arbitrary $\lambda$-term is $\beta$-equivalent to at most one term in normal form. Give an example of a $\lambda$-term having no normal form.

**2** Explain what is meant by a CL-term (or combinator term) in a given set $X$ of free variables, and by the statement that two CL-terms are weakly equivalent. Show how to associate with each $\lambda$-term $M$ a CL-term $M^*$, with the same set of free variables, in such a way that $x^* = x$ for each variable $x$, $(MN)^* = M^*N^*$ for all $M$ and $N$, and $M$ and $N$ are $\beta$-equivalent whenever $M^*$ and $N^*$ are weakly equivalent. By considering the $\lambda$-terms $\lambda x.x$ and $\lambda x.((\lambda y.y)x)$, or otherwise, show that the converse of the last statement fails.

[You may assume the Church–Rosser property for weak equivalence of CL-terms.]

**3** (*a*) Explain what is meant by a c.p.o. (with least element) and by a continuous map of c.p.o.'s. If $D$ and $D'$ are c.p.o.'s, show that the set $[D \to D']$ of continuous maps $D \to D'$ can be given the structure of a c.p.o. Deduce that the category of c.p.o.'s and continuous maps between them is cartesian closed.

(*b*) Let $\mathcal{C}$ be a cartesian closed category, and $D$ an object of $\mathcal{C}$ satisfying $D^D \cong D$. Explain briefly how the set of morphisms $D \to D$ in $\mathcal{C}$ may be made into a model of the (untyped) $\lambda$-calculus satisfying the $\beta$- and $\eta$-rules (i.e., such that $\beta\eta$-equivalent terms have the same interpretation).

**4** Explain what is meant by an embedding–projection pair of continuous maps between c.p.o.'s. Given a c.p.o. $D_0$, define $D_n = [D_{n-1} \to D_{n-1}]$ for $n > 0$; show that if we are given an embedding–projection pair $(\phi_0 \colon D_0 \to D_1, \psi_0 \colon D_1 \to D_0)$, we may obtain such pairs $(\phi_n \colon D_n \to D_{n+1}, \psi_n \colon D_{n+1} \to D_n)$ for all $n$ by setting

$$\phi_n(f) = \phi_{n-1} \circ f \circ \psi_{n-1} \quad , \quad \psi_n(g) = \psi_{n-1} \circ g \circ \phi_{n-1} \ .$$

Explain briefly how this result may be used to construct a c.p.o. $D_\infty$ satisfying $D_\infty \cong [D_\infty \to D_\infty]$, equipped with embedding–projection pairs $(D_n \to D_\infty, D_\infty \to D_n)$ for all $n$.

**5** Let $\mathcal{C}$ be a cartesian closed category having just two objects 1 (the terminal object) and $D$, and suppose $\mathcal{C}$ is not a preorder. Show that we necessarily have $D \times D = D^D = D$, and that the monoid $M$ of morphisms $D \to D$ in $\mathcal{C}$ comes equipped with distinguished elements $\pi, \pi'$ and $\epsilon$, a unary operation $(-)^*$ and an additional binary operation $\langle -, - \rangle$ satisfying

$$\pi\langle x, y\rangle = x \quad , \quad \pi'\langle x, y\rangle = y \quad , \quad \langle \pi z, \pi' z\rangle = z \quad ,$$

$$\epsilon\langle x^*\pi, \pi'\rangle = x \quad \text{and} \quad (\epsilon\langle y\pi, \pi'\rangle)^* = y$$

for all $x, y, z \in M$. Conversely, given a monoid $M$ with this additional structure, show that the element $(\pi')^*$ is idempotent, and hence construct a two-object cartesian closed category such that $M$ appears as the monoid of endomorphisms of its non-terminal object.

## END OF PAPER