UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 9 to 12

PAPER 6

INTRODUCTION TO BANACH SPACES AND ALGEBRAS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let E be a Banach space, F a normed space and let T be a bounded linear mapping from E to F. Suppose that there exist constants $R > 0$ and $0 < k < 1$ such that, for every $y \in F$ with $\|y\| \leqslant 1$, there is some $x \in E$ with $\|x\| \leqslant R$ such that $\|Tx - y\| \leqslant k$. Prove that (i) for every $y \in F$ there is some $x \in E$ with $Tx = y$ and $\|x\| \leqslant \left(\frac{R}{1-k}\right)\|y\|$ and (ii) F is complete.

State, *without proof*, the Open Mapping Theorem, for a continuous linear map between two Banach spaces. Deduce the Closed Graph Theorem.

Let E, F be Banach spaces and let $T : E \to F$ be a linear mapping. Define a subset S of F by saying that an element y of F belongs to S if and only if there is some sequence (x_n) in E such that $x_n \to 0$ in E, while $Tx_n \to y$ in F. Prove that S is a closed linear subspace of F, and that T is continuous if and only if $S = \{0\}$.

2 Let E be a real vector space and let K be a convex subset of E. Define what it means to say that a point $x \in K$ is an *extreme point* of K.

Now let E be a real, Hausdorff, locally convex topological vector space and let K be a compact, convex subset of E. Prove that K is the closed convex hull of its set of extreme points. [*You may assume, without proof, any version of the Hahn-Banach theorem, and of the separation theorem for convex sets.*]

Define the standard real Banach spaces ℓ^1 and ℓ^∞, and identify the extreme points of their closed unit balls. [*You should assume, without proof, that these spaces* are *Banach spaces*.]

3 Let A be a Banach algebra, with identity, over the complex field, and let $x \in A$. Define the *spectrum*, $\operatorname{Sp} x$, of x in A. Prove that $\operatorname{Sp} x$ is a non-empty, compact subset of $\mathbb{C}$. [*You should assume, without proof, that*: (i) *the set $G(A)$ of invertible elements of A is open in A, and* (ii) *the mapping $x \mapsto x^{-1}$ $(x \in G(A))$ is continuous*.]

Let $E = \mathcal{O}(\mathbb{C})$ be the algebra of all complex-valued, holomorphic (i.e. analytic) functions on the complex plane. Show that E can be given an algebra-norm, but not a complete algebra-norm.

Let $D = C(\mathbb{C})$ be the algebra of all complex-valued, continuous functions on the complex plane. Explain, briefly, how to find, in D, a function f and a sequence (g_n) of non-identically-zero functions, such that $fg_n = ng_n$, for all $n \geqslant 1$. Deduce that D cannot be given any algebra-norm.

4 Let A be a commutative Banach algebra, with identity, over the complex field. Let $x \in A$, let U be an open neighbourhood of $\operatorname{Sp} x$ in $\mathbb{C}$ and let $\mathcal{O}(U)$ be the algebra of all complex-valued, holomorphic functions on U, in its standard topology (i.e. local uniform convergence). Prove that there is a unique, continuous, unital homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(Z) = x$ (where Z is the coordinate function $Z(\lambda) = \lambda$ on U). [*You may assume, without proof, any form of the Runge approximation theorem.*]

Prove also that, for every character ϕ on A and every $f \in \mathcal{O}(U)$, we have $\phi\big(\Theta_x(f)\big) = f\big(\phi(x)\big)$.

An element e of A is called *idempotent* if and only if $e^2 = e$. Suppose that A contains an element a such that $\operatorname{Sp} a = X \cup Y$, where X, Y are disjoint, non-empty, closed subsets of $\mathbb{C}$. Prove that A contains an idempotent e such that, for every character ϕ, $\phi(e) = 1$ whenever $\phi(a) \in X$ and $\phi(e) = 0$ whenever $\phi(a) \in Y$.

5 Give an account of the elementary theory of C^*-algebras, leading to a proof of the Gelfand-Naimark theorem for commutative C^*-algebras.

Let x be a normal element of a C^*-algebra A. Prove that there is a unique continuous, unital $*$-homomorphism $\theta_x : C(\operatorname{Sp} x) \to A$ such that $\theta_x(Z) = x$ (where Z is the coordinate function $Z(\lambda) = \lambda$ on $\operatorname{Sp} x$). Show, also, that θ_x is isometric, and that the image of θ_x is the closed subalgebra of A generated by $\{x, x^*\}$.

Prove that there is a unique hermitian element h of A such that both $h^2 = x^*x$ and $\operatorname{Sp} h \subset \mathbb{R}^+$.