UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 9.00 to 12.00

PAPER 38

INTERACTING PARTICLE SYSTEMS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $G = (V, E)$ be a finite connected graph with $|E| \geq 1$, considered as an electrical network with strictly positive conductances w_e, $e \in E$, and source s, sink t. State Kirchhoff's first and second laws for the currents and potential differences of the network. State Ohm's law.

Let $X = (X_n : n \geq 0)$ be a Markov chain on the state space V with transition matrix

$$p_{xy} = \frac{w_e}{\sum_{f \sim x} w_f}$$

where e is the edge $\langle x, y \rangle$ and the summation is over all edges f incident to x. Thus $p_{xy} = 0$ if either $x = y$ or x is not a neighbour of y. The chain starts at $X_0 = s$, and it stops at the first time it visits t.

Let u_{xy} be the expected total number of one-step transitions of the chain from x to y; each transition from x to y counts $+1$, and from y to x counts -1. Show that u satisfies the two Kirchhoff laws with a total flow of one, and deduce that u_{xy} equals the current along $\langle x, y \rangle$ from x to y when the total flow equals one.

[*A clear statement should be given of any general result to which you appeal.*]

2 Let $\Omega = \{0, 1\}^E$ where E is a finite set. Define an *increasing* subset of Ω. For increasing subsets A, B of Ω, define the subset $A \circ B$ [sometimes written $A \,\square\, B$] containing vectors $\omega \in \Omega$ for which 'A and B occur disjointly'.

State the BK 'disjoint-occurrence' inequality for the product measure P_p on Ω with density p.

Consider bond percolation on $\mathbb{Z}^2$ with density p, and let A be the event that there exists an open path that crosses the rectangle $[0, 2n] \times [0, 2n - 1]$ from its left side to its right side. Show that

$$P_p(A) + P_{1-p}(A) = 1.$$

By considering the open clusters at vertices of the form (n, y) for $0 \leq y \leq 2n$, show that

$$P_{\frac{1}{2}}(\mathrm{rad}(C) \geq n) \geq \frac{1}{2\sqrt{n}}$$

where C is the cluster at the origin 0 and $\mathrm{rad}(C) = \max\{n : 0 \leftrightarrow \partial\Lambda_n\}$ with $\Lambda_n = [-n, n]^2$.

3 Let $G = (V, E)$ be a finite graph. Let $p \in (0,1)$, $q \in \{2,3,\ldots\}$, and write $\Omega = \{0,1\}^E$ and $\Sigma = \{1,2,\ldots,q\}^V$. Let κ be the probability measure on $\Omega \times \Sigma$ given by

$$\kappa(\omega,\sigma) = \frac{1}{Z}\prod_{e\in E}\Big\{(1-p)\delta_{\omega(e),0} + p\delta_e(\sigma)\delta_{\omega(e),1}\Big\},$$

where $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ for $e = \langle x, y\rangle \in E$.

Show that the first marginal measure of κ is the random-cluster measure $\phi_{p,q}$, and the second marginal measure is the Potts measure $\pi_{\beta,q}$, where $p = 1 - e^{-\beta}$. Derive the conditional measure on Ω given the vertex-configuration σ, and the conditional measure on Σ given the edge-configuration ω.

Prove that

$$\left(1-\frac{1}{q}\right)\phi_{p,q}(x \leftrightarrow y) = \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q},$$

and explain how this can be used to relate the phase transitions of the random-cluster and the Potts models.

4 Let $\Omega = \{0,1\}^E$ where E is a finite set, and let μ_1 and μ_2 be probability measures on Ω. Explain what is meant by saying that μ_1 dominates μ_2 stochastically. State the Holley condition for this to occur.

Let $G = (V, E)$ be a finite graph, and let $\phi_{p,q}$ be the random-cluster measure on G with parameters p and q. Prove that

$$\phi_{p',1} \le_{\rm st} \phi_{p,q} \le_{\rm st} \phi_{p,1}, \qquad q \ge 1,\ p \in (0,1),$$

where $p' = p/[p + q(1-p)]$ and $\le_{\rm st}$ denotes stochastic ordering. [You may need the fact that $k(\omega) + \eta(\omega)$ is a non-decreasing function of ω, where $k(\omega)$ is the number of open clusters of ω and $\eta(\omega)$ is the number of open edges.]

Let $p_{\rm c}(q)$ denote the critical point of the (wired) random-cluster measure on $\mathbb{Z}^d$, where $q \ge 1$. Show that $p_{\rm c}(1) \le p_{\rm c}(q)$, and derive an upper bound for $p_{\rm c}(q)$ in terms of $p_{\rm c}(1)$.

5 Describe the graphical representation of the contact model on $\mathbb{Z}^d$, in terms of Poisson processes of deaths and infection with respective intensities δ and λ. Let $\delta = 1$, and write $\theta(\lambda)$ for the probability that an initial infection at the origin persists in $\mathbb{Z}^d$ for all future time. Explain why θ is a non-decreasing function, and define the critical point $\lambda_c = \lambda_c(d)$ of the process.

Prove that $\lambda_c \geq (2d)^{-1}$.

By coupling the d-dimensional process with the one-dimensional process with infection rate $d\lambda$, or otherwise, show that $\lambda_c(d) \leq d^{-1}\lambda_c(1)$.

6 Let $G = (V, E)$ be a finite graph, and $\Sigma = \{-1, +1\}^V$. For $x \in V$, $\sigma \in \Sigma$, let $N(x) = \Sigma_{y \sim x}\sigma_y$ be the sum of the states of the neighbours of x. Consider a discrete-time Markov chain $X = (X_n : n \geq 0)$ on Σ with transition probabilities

$$p(\sigma_x, \sigma^x) = \frac{1}{|V|} \cdot \frac{e^{2N(x)}}{1 + e^{2N(x)}},$$
$$p(\sigma^x, \sigma_x) = \frac{1}{|V|} \cdot \frac{1}{1 + e^{2N(x)}},$$

for $x \in V$, $\sigma \in \Sigma$, where σ_x (respectively, σ^x) is the configuration obtained from σ by setting the value -1 (respectively, $+1$) at the position labelled x. Let $p(\sigma, \sigma') = 0$ if σ, σ' differ at more than one vertex. Show that X is a (time-)reversible Markov chain with respect to the Ising measure

$$\pi(\sigma) = \tfrac{1}{Z} \exp\left(\sum_{x \sim y} \sigma_x \sigma_y\right), \quad \sigma \in \Sigma,$$

where the summation is over all unordered pairs of neighbouring vertices.

Explain how the chain X may be used in a system of 'coupling from the past' in order to generate a random sample with measure π. Prove that the 'coupling from the past' algorithm terminates in finite time, with probability one.

END OF PAPER