UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 1.30 to 4.30

PAPER 16

GLOBAL RIEMANNIAN GEOMETRY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let M be a Riemannian manifold. Define Jacobi field, conjugate point and the mutiplicity of a conjugate point.

(b) Show that the multiplicity of a conjugate point cannot exceed $n-1$, where n is the dimension of M.

(c) Let M be a Riemannian manifold, $\gamma : [0,1] \to M$ a geodesic and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$ such that $f(0,t) = \gamma(t)$, the curves $t \mapsto f(s,t)$ are geodesics and $J(t)=\frac{\partial f}{\partial s}(t,0)$.

2 (a) Let $f_i : M \to N$, $i = 1, 2$ be two local isometries between connected Riemannian manifolds. Show that if there exists $p \in M$ such that $f_1(p) = f_2(p)$ and $(df_1)_p = (df_2)_p$, then $f_1(q) = f_2(q)$ for all $q \in M$.

(b) Let M be a complete Riemannian manifold with constant sectional curvature $K \equiv 1$. Show that the universal covering of M with the induced metric is isometric to S^n. [*You may assume Cartan's theorem.*]

(c) Let M be an even dimensional complete Riemannian manifold with constant sectional curvature $K \equiv 1$. Show that M is isometric to S^n or to the real projective space of dimension n. Is the same result true in odd dimensions?

3 (a) Let $\gamma : [0,\infty) \to M$ be a unit speed geodesic of a complete Riemannian manifold M. Show that γ cannot minimize distance past its first conjugate point. [*You may assume the formula for the second variation of energy.*]

(b) Let $\gamma(t_0)$ be the cut point of $p = \gamma(0)$ along γ. Show that either $\gamma(t_0)$ is the first conjugate point of p along γ or there exists a geodesic σ different from γ, joining p to $\gamma(t_0)$, and such that σ also minimizes the distance from p to $\gamma(t_0)$.

(c) Give an example of a positively curved manifold M and a point $p \in M$ such that for any geodesic γ with $\gamma(0) = p$, the cut point of p along γ does not coincide with the first conjugate point of p along γ.

4 Let M^n be a complete manifold with $\mathrm{Ric} \geqslant k > 0$.

(a) State the Bonnet-Myers theorem.

(b) Let S^n_k be the n-dimensional sphere with constant curvature k. Using the classification theorem of complete manifolds of constant sectional curvature, show that if $\mathrm{Vol}(M) = \mathrm{Vol}(S^n_k)$, then M is isometric to S^n_k.

(c) Show that if $\mathrm{diam}(M) = \pi/\sqrt{k}$, then M is isometric to S^n_k.

[*You may use comparison results for Ricci curvature, but they should all be clearly stated.*]

5 (a) Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geqslant 0$. Let Γ be any finitely generated group of isometries acting properly discontinuously on M. Prove that Γ has polynomial growth of degree less than or equal to n. [*You may use a comparison result for Ricci curvature.*]

(b) Let M^n be a compact Riemannian manifold with $\mathrm{Ric} \geqslant 0$. Show that the first Betti number of M is less than or equal to n.

(c) By using (a) or otherwise exhibit a connected Lie group which does not admit a bi-invariant metric. [*You may assume that bi-invariant metrics on a Lie group have non-negative sectional curvature.*]