

PAPER 28

GEOMETRY OF 3-DIMENSIONAL MANIFOLDS

*Attempt **THREE** questions.*

*There are **five** questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions  
printed on the subsequent pages until  
instructed to do so by the Invigilator.**

**1** Let  $A$  be a finitely generated free abelian group of rank  $n$ , i.e.  $A$  is isomorphic to the direct product of  $n$  copies of the integers  $\mathbb{Z}$ . For what values of  $n$  is  $A$  isomorphic to the fundamental group of a connected, compact, orientable 3-manifold without boundary? How unique is the manifold in each case?

Describe the six flat orientable 3-manifolds.

**2** Define the terms nilpotent group and polycyclic group. Indicate briefly how to prove that a finitely generated nilpotent group is polycyclic. Show that, up to finite index, a polycyclic  $PD^3$ -group is an extension of a normal abelian subgroup of rank 2 by an infinite cyclic group. Under what conditions is this group nilpotent?

If  $\text{Nil}^3$  is the nilpotent Lie group consisting of all real matrices of the form

$$X = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and  $k$  is a positive integer, let  $N_k$  be the discrete subgroup consisting of all matrices for which  $x, y$  and  $z$  are integers divisible by  $k$ .

Show that  $X \mapsto (x \bmod k, y \bmod k)$  is the projection map from  $\text{Nil}^3/N_k$  onto the base  $T^2$  of a locally trivial  $S^1$ -fibration. Prove further that

$$H_1(\text{Nil}^3/N_k, \mathbb{Z}) \cong N_k/N'_k \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/k.$$

**3** Explain the use of the spectral sequence of the group extension

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

in the proof that, if  $N$  is a  $PD^n$ -group and  $Q$  is a  $PD^q$ -group, then  $G$  is a  $PD^{n+q}$  group.

Suppose now that  $N$  is a normal, finitely presented subgroup of the  $PD^3$ -group  $G$  and that  $G/N$  contains elements of infinite order. Show that either  $N$  is a  $PD^2$ -group, and hence a surface group, or that  $N$  is free. In the first case deduce that  $G$  contains a subgroup of finite index, which is the fundamental group of a surface fibration over  $S^1$ .

**4** Outline the main steps in the proof of the Loop Theorem.

Let  $N$  be the closure of the complement of a solid torus neighbourhood of a knot  $K$  in  $S^3$ . If  $K$  is not unknotted, show that the inclusion  $i: \partial N \rightarrow N$  induces an injection  $i_*: \pi_1(\partial N, *) \rightarrow \pi_1(N, *)$ . (Here  $*$  denotes some base point chosen to lie in  $\partial N$ .)

Deduce that a knot  $K$  is the unknot if and only if  $\pi(S^3 - K, *)$  is infinite cyclic.

**5** Write an essay on the isometry group  $I(M)$  of a Riemannian manifold, paying special attention to dimension three.