UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 1.30 to 4.30

PAPER 40

GALAXIES

*Answer **THREE** questions. The questions are of equal weight.*

*Questions for which **both** parts are completed score more highly than two partial answers.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) **Derive** the equation for the chemical evolution of a galaxy in the form

$$\frac{d}{ds}(gZ_i) = -\frac{1-\beta f}{1-\beta} Z_i + y_i f,$$

where

$g(s)$ is the interstellar mass in gas and dust,

s is the mass in stars,

β is the constant fraction of the mass returned to the interstellar medium per unit mass initially formed into stars,

y_i is the constant nucleogenic yield of element i per unit mass remaining in stars,

$Z_i(s)$ is the interstellar abundance of element i,

f is the fraction of the returned mass retained in the galaxy.

Assuming the above equation is solved for $Z_i(s)$, and hence $s(Z_i)$, **explain** how this function gives information on the relative numbers of stars that have different metal abundances.

(ii) If s_∞ is the final mass in stars when the gas is exhausted and $G = g/s_\infty, S = s/s_\infty$ and the function $G(S)$ is given by $G = S(1-S)$ **show** that even if f varies

$$\frac{d}{ds}(GZ_i) + \frac{(1-\beta f)GZ_i}{(1-\beta)S(1-S)} = y_i f.$$

When $f = S$ use an integrating factor to **show** that

$$Z_i = \frac{y_i}{S^q} \int_o^s \frac{S^q}{1-S}\, dS = y_i \sum_1^\infty \frac{S^n}{q+n} \text{ where } q = \frac{2-\beta}{1-\beta} \ .$$

2 (i) Jeans's equation of stellar hydrodynamics for the number density $n(\mathbf{r})$ and the mean velocity $\mathbf{u}(\mathbf{r})$ of stars of a particular type is

$$n(\mathbf{r})\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\nabla)\mathbf{u}\right] = -div\mathbf{p} + n\nabla\psi,$$

where $\mathbf{p}$ the 'pressure tensor' is defined in terms of the distribution function $f(\mathbf{v},\mathbf{r})$ for the stars of that type by $\mathbf{p} = \int f(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})d^3v$.

Show that in a stationary system in which $\mathbf{u} = 0$ and f is isotropic in $\mathbf{v}$, Jeans's equation reduces to the hydrostatic equation for the scalar 'pressure' p.

$$\nabla p = n\nabla\psi.$$

Such a system is observed to be spherical with a projected number density of these stars $N(R)$ at projected distance R from the centre and a one dimensional projected Döppler velocity dispersion $\sigma(R)$ in the line of sight there. Show that $n(r)$ and $p(r)$ are related to the observed $N(R)$ and $\sigma(R)$ by

$$n(r) = -\frac{1}{2\pi r}\frac{d}{dr}\left[\int_{r^2}^{\infty}\frac{N(R)}{\sqrt{R^2-r^2}}dR^2\right] \text{ and } p(r) = \frac{-1}{2\pi r}\frac{d}{dr}\left[\int_{r^2}^{\infty}\frac{N(R)\sigma^2(R)}{\sqrt{R^2-r^2}}dR^2\right]$$

[*You may assume* $\int_a^b \frac{dt}{\sqrt{(t-a)(b-t)}} = \pi$.]

(ii) If $N(R) = N_o b^5/(R^2+b^2)^{5/2}$ and $\sigma(R) = \sigma_o b^{1/2}/(R^2+b^2)^{1/4}$ where N_o, σ_o and b are constants, use the integration variable $s = \frac{R^2+b^2}{r^2+b^2}$ to show that $n(r) = \frac{2}{\pi}\frac{N_o b^5}{(r^2+b^2)^3} I_{5/2}$ where $I_n = \int_1^\infty s^{-n}(s-1)^{-1/2}ds$, and find a similar expression for $p(r)$. Given that $I_{5/2} = 4/3$ and $I_3 = \frac{3\pi}{8}$, use Part (i) to give the total gravitational acceleration at $\mathbf{r}$ as

$$\nabla\psi = -\frac{\mathbf{r}}{(r^2+b^2)^{3/2}}\,\sigma_o^2\, b\;\; \frac{315\pi}{128}\;.$$

How may the mass density of **all** the matter present be found from this?

3 (i) The gravitational potential of a point mass, m, at distance b below the origin is given by

$$\psi_I(R,z) = Gm/\sqrt{R^2+(z+b)^2}$$

or by $\psi_{II} = \psi_I - Gmb^{-1}$ if the potential is zeroed at the origin. **Calculate** the gravitational flux per unit area through the plan $z = 0$ at distance R from the origin. Use Kuzmin's reflection method to find the surface density $\Sigma(R)$ that gives the above potential for $z \geqslant 0$ and its reflection for $z \leqslant 0$.

Find as an integral the surface density that likewise gives the potential

$$\psi_I(R,z) = G\int_0^\infty \frac{\mu(b)db}{[R^2+(|z|+b)^2]^{1/2}}$$

$$\text{or}\quad \psi_{II} = G\int_0^\infty \left(\frac{1}{[R^2+(|z|+b)^2]^{1/2}} - \frac{1}{b}\right)\mu(b)db$$

(ii) If $\mu(b) = Bb^\beta$ with $0 < \beta < 1$ **show** that ψ_I diverges at the origin but that ψ_{II} remains *zero* there. Calculate $\partial\psi_{II}/\partial R$ on $z = 0$ and use the substitution $b^2 = R^2\frac{t}{1-t}$ to show that $\partial\psi/\partial R = -CR^{\beta-1}$ there

where $C = \frac{1}{2}GB\int_0^1 t^{(\beta-1)/2}(1-t)^{-\beta/2}dt = \frac{1}{2}GB\frac{\Gamma(\frac{\beta+1}{2})\Gamma(\frac{2-\beta}{2})}{\Gamma(\frac{3}{2})}$.

[*You may quote the result that* $\int_0^1 t^{z-1}(1-t)^{w-1}dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$.]

Hence use Part (i) to **find** the surface density of a flat galaxy whose circular velocity is $V = KR^{\beta/2}$ where K is a known constant.

4 (i) Schwarzschild's metric for a spherical black hole is

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) \; ;$$

where $m = \frac{GM}{c^2}$.

A steady equatorial accretion disk carries a rest-mass flux $F = 2\pi r \Sigma u$ inwards into the hole where $u = -dr/d\tau$ and τ is the proper time measured on a circulating fluid element. If $h(r)$ is the specific angular momentum of the circular orbit of radius r and $\Omega(r)$ is its angular velocity as seen from infinity, **show** that, if no angular momentum is radiated, the inward radial component of the 4-velocity of the disk at r is

$$u = \nu r^2 \left(\frac{-d\Omega}{dr}\right) / (h - h_o)$$

where ν is the kinematic viscosity of the disk and h_o is the specific angular momentum swallowed by the black hole.

Given that two first integrals of the geodesic equation of a free particle moving in the equatorial plane $\theta = \pi/2$ are

$$\left(1 - \frac{2m}{r}\right)\frac{dt}{d\tau} = \epsilon / c^2,$$

and

$$r^2 \frac{d\phi}{d\tau} = h,$$

and that by definition of $d\tau$ in an equatorial orbit

$$c^2 = \left(1 - \frac{2m}{r}\right)c^2\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2 ,$$

show that

$$\frac{d^2 r}{d\tau^2} = -\frac{mc^2}{r^2} + \frac{h^2}{r^3}\left(1 - \frac{3m}{r}\right) ,$$

hence **show** that circular orbits have

$$h(r) = \sqrt{\frac{m}{r - 3m}} \;\; r\,c$$

and the minimum h is $h_0 = \sqrt{12}\;\; mc$, at $r = 6m$.

(ii) A fluid element of the disk is observed from afar circulating down from $r = r_1$ to $r = 6m$. Given that $\Omega(r) = c\sqrt{m/r^3}$ **show** that the time it is seen to take is approximately *

$$\Delta t = \frac{2}{3\nu}\int_{6m}^{r_1}\left[r - \sqrt{12m(r - 3m)}\right]\frac{r\;dr}{(r - 3m)} =$$

$$\frac{m^2}{\nu}\left[3X^2 - 8(X-1)^{3/2} + 6X - 24(X-1)^{1/2} - 6\;ln(X-1) + 8\right]$$

where $X = r_1/(3m)$.

* The light travel time across the disk is to be neglected and $(dr/d\tau)^2$ is neglected as compared to h^2/r^2 in evaluating $dt/d\tau$.