UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 6 June, 2005 1.30 to 4.30

PAPER 80

FUNDAMENTALS OF ATMOSPHERE-OCEAN DYNAMICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Clarity and explicitness of reasoning will attract more credit than perfection of computational detail

(x, y, z) *denotes right-handed Cartesian coordinates and* (u, v, w) *the corresponding velocity components;* t *is time; the gravitational acceleration is* $(0, 0, -g)$ *where* g *is a positive constant;* $\hat{z} = (0, 0, 1)$ *is a unit vector directed vertically upward.*
The fluid is always incompressible. 'Ideal fluid' always means that buoyancy diffusion can be neglected where relevant, as well as viscosity. N *denotes the buoyancy frequency of a stratified fluid.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write down the Boussinesq momentum, mass-continuity and buoyancy equations for a continuously stratified, non-rotating, ideal fluid. Buoyancy effects should be represented in the equations by a suitably defined buoyancy acceleration $\sigma(\mathbf{x},t)$, to be specified. Define the buoyancy frequency $N(z)$. State briefly the conditions under which the Boussinesq approximation is valid, given the incompressibility of the fluid.

For 2-dimensional motion with $\partial/\partial y = 0$, derive the y-component of the corresponding vorticity equation in the form

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial\sigma}{\partial x} \;=\; \left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2\psi \;,$$

where $\psi(x,z,t)$ is a streamfunction to be specified. For small disturbances, linearize this equation and the buoyancy equation about a basic state in which the flow is in the x direction with velocity $\bar{u}(z) > 0$, and in which $N = N(z) > 0$. Using subscripts to denote partial derivatives, show from the linearized equations that

$$D_t^2\nabla^2\psi' \;-\; \bar{u}_{zz}D_t\psi'_x \;+\; N^2\psi'_{xx} \;=\; 0,$$

where ψ' and σ' are the disturbance streamfunction and buoyancy acceleration, respectively, and D_t is the linearized material derivative, $D_t = \partial/\partial t + \bar{u}\,\partial/\partial x$. Deduce that stationary wave disturbances of the form $\psi' = \widehat{\psi}(z)\,\exp(ikx)$, real part understood, k real, satisfy

$$\widehat{\psi}_{zz} \;+\; m^2(z)\,\widehat{\psi} \;=\; 0$$

where

$$m^2(z) \;=\; \frac{N^2(z)}{\bar{u}^2(z)} - \frac{\bar{u}_{zz}}{\bar{u}} - k^2 \;=\; \ell^2(z) - k^2 \quad \text{say.}$$

Briefly discuss the qualitative nature of such disturbances when a rigid lower boundary at $z = 0$ is present, but no upper boundary. On the assumption that $\bar{u}(z)$ increases linearly with altitude z while $N^2(z)$ is large enough but bounded, explain qualitatively why, for certain values of k, the disturbances are trapped against the lower boundary.

A rigid upper boundary $z = H > 0$ (H constant) is now introduced. Both N and $\bar{u}$ are made constant. Find solutions for the possible stationary wave disturbances with real k when $2\pi < NH/\bar{u} < 3\pi$, giving expressions for the ψ' and σ' fields. Deduce that the most general possible superposition of such solutions involves just two complex arbitrary constants.

Show that for all values of the two constants, regardless of their magnitude, the superposition just found satisfies not only the linearized equations but also the full nonlinear equations — i.e. that it is a finite-amplitude solution.

2 Write down the full shallow-water equations, for uniform undisturbed depth h_{00} say, in a frame of reference rotating with constant angular velocity $(0, 0, \frac{1}{2}f)$. Give the order-of-magnitude analysis that leads from the full shallow-water equations to the quasi-geostrophic shallow-water equations, explaining carefully the scaling assumptions used. Briefly explain the concepts of 'potential-vorticity conservation' and 'potential-vorticity inversion'. Define the Rossby length L_{R} and explain how it enters the inversion problem.

Briefly comment on the implications of the single time derivative $\partial/\partial t$ that appears in the quasi-geostrophic theory, as compared with the three such derivatives in the full shallow-water equations. What types of motion described by the full equations are excluded from the quasi-geostrophic equations?

Uniform flow $\mathbf{u} = (U, 0)$ in a quasi-geostrophic shallow-water system with uniform potential vorticity encounters topography such that the layer depth with an undisturbed free surface would be $h_{00} - b(x, y) > 0$. This results in a steady flow pattern with disturbance streamfunction $\psi'(x, y)$, representing the topographically-induced departure from uniform flow. Show that a disturbance streamfunction $\psi' \propto \exp\{-(x^2 + y^2)/a^2\}$, where a is a constant length scale, provides a solution to this problem in the case where the topography $b(x, y)$ has the circularly symmetric shape

$$z = b(x, y) = \epsilon \left\{ \frac{4(a^2 - x^2 - y^2)}{a^2} + \frac{a^2}{{L_{\mathrm{R}}}^2} \right\} \exp \left\{ \frac{-(x^2 + y^2)}{a^2} \right\} ,$$

where ϵ is a sufficiently small constant. Discuss (a) how small ϵ must be in order for the solution to represent a self-consistent use of quasi-geostrophic theory, and (b) how small ϵ must be if the flow over the topography is to have no closed streamlines.

Discuss briefly the extent to which the appearance of closed streamlines in the solution might, or might not, affect the validity of the solution as a model of physical reality.

3 For a Boussinesq, stably stratified ideal fluid on a rotating Earth, derive the equation governing the quasi-geostrophic potential vorticity

$$Q \;=\; f + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) \;,$$

taking as your starting point the buoyancy and vorticity equations in the form

$$\frac{D_{\rm g}}{Dt}\left(f_0\,\frac{\partial\psi}{\partial z}\right) + N^2 w = 0$$

and

$$\frac{D_{\rm g}}{Dt}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] + \beta\,\frac{\partial\psi}{\partial x} \;=\; f_0\,\frac{\partial w}{\partial z} \;,$$

with $f = f_0 + \beta y$, in the notation of the lecture notes. Do **not** give the full order-of-magnitude analysis, but briefly explain the meaning of the symbols ψ and $\dfrac{D_{\rm g}}{Dt}$, showing why the buoyancy acceleration $\propto \partial\psi/\partial z$. Through what approximate balance of terms is ψ related to the velocity field? Briefly state the conditions on dimensionless parameters that tend to favour such balance, and briefly explain why only the vertical component of the Earth's rotation rate enters the dynamics. Show that for a nearly-horizontal lower boundary with small elevation $z = b(x, y)$ the boundary condition on ψ is

$$\frac{D_{\rm g}}{Dt}\left(\frac{f_0}{N^2}\,\frac{\partial\psi}{\partial z} + b\right) = 0 \quad \text{at} \quad z = 0 \;.$$

Derive the dispersion relation for Rossby waves of form $\exp i(kx + mz - \omega t)$, as described by the linearized quasi-geostrophic equations, in the case where the waves propagate on a background state of relative rest and constant N. Sketch the Q field at a fixed altitude z, and describe the physical mechanism giving rise to the phase propagation in the x direction.

Derive the group velocity and show that its vertical component is always upward when the vertical phase velocity is downward, and *vice versa*.

Find the stationary disturbance pattern generated when the fluid is no longer at relative rest, but flows with uniform velocity $(U,\,0,\,0)$ over a corrugated boundary with $b = b(x) = b_0\sin(kx)$ where b_0 and k are real constants, and $U \neq \beta/k^2$. The fluid is unbounded above. Carefully explain the reasoning necessary to make the solution unique.

Comment briefly on the significance of this solution to our understanding of the dynamics of the stratosphere.

4 Write an essay on the effects of stable stratification upon ideal fluid flow. You may **either** concentrate on stratification alone, illustrating your main points through selected examples, **or** take a broader view, with fewer examples, touching on how the phenomena are changed by Coriolis effects. A good starting point might be to discuss briefly why a spherical blob of fluid displaced vertically does not oscillate with frequency N.

END OF PAPER