UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2008 1.30 to 4.30

PAPER 84

FLUID DYNAMICS OF ENERGY

*Attempt no more than **THREE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

*This is an **OPEN BOOK** examination.*

Candidates may bring handwritten notes and lecture handouts into the examination.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Fluid is injected into a deep saturated horizontal confined porous layer of thickness w, permeability k and porosity ϕ from a line source $x = 0$. The injected fluid has density $\rho + \Delta\rho$ while the original fluid in the porous layer has density ρ, and both fluids have the same viscosity μ. The lower boundary of the layer consists of a thin low permeability layer of rock with permeability $\lambda < k$ and thickness b. Beneath this layer, there is a region of much higher permeability, so that fluid can migrate across the thin layer to a region with constant pressure, and the cross-layer flow is associated with the excess hydrostatic pressure of the injected fluid on the low permeability layer.

(a) Derive the equation to describe the shape of the advancing interface between the two fluids:

$$\frac{\partial h}{\partial t} = \frac{k\Delta\rho g}{\phi\mu}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) - \frac{\lambda k\Delta\rho g}{\mu b}\,h\,.$$

(b) If a finite volume of fluid V is released into the aquifer, find a solution of the above equation which describes how the volume and the lateral extent of the current evolve with time.

[*Hint : transform the equation from the variable* $h(x,t)$ *to* $H(x,t) = h\,\exp(\Omega\,t)$, *and then rescale time.*]

(c) Use the solution to calculate the maximum lateral extent of the current.

(d) Also, as the current advances, calculate the lateral position at which the depth of the current is instantaneously constant, and hence find an expression for the locus of the region of rock invaded by the injected fluid.

2 Fluid is injected into a porous layer composed of two parallel layers of different permeability, k_1 and k_2, and porosity ϕ from a point $x = 0$ to a point $x = L$. The layers are bounded by impermeable seal rock above and below and are in very poor contact with each other.

(a) If a pressure difference ΔP is imposed between the points $x = 0$ and $x = L$, calculate the time of arrival t_a of the injected fluid at the distal point $x = L$, where x is the along-layer coordinate, and explain how the flux of injected fluid at $x = L$ varies with time for $t > t_a$. In particular, at what time t_s does the injected fluid emerge from the lower permeability layer?

(b) In the next pair of wells, in order to reduce the flow of the injected fluid at the point $x = L$, a slug of polymer is added to the injected fluid, over a time Δt, as soon as some of the fluid being injected is recorded as arriving at the outflow well, $x = L$. The polymer is designed to activate at time τ after entering the rock, and at this time it increases the viscosity of the fluid by a factor λ. Develop a model to describe how the flow of injected fluid changes with time, for $t > \tau$, and in particular to describe the partitioning of the inflow fluid between the high and low permeability layers, and the variation with time of the overall flux.

(c) Determine whether there are conditions under which this strategy leads to a greater flux of original fluid being produced from the reservoir at a given time after the start of the injection.

3 Radioactive fluid migrates through a laterally extensive aquifer with permeability $k(z) = k_0 z(h - z)$ where z denotes the depth in the aquifer.

(a) Derive an equation to describe the dispersion of the radioactive tracer as it spreads through the rock, if a line source injects the fluid at a constant rate Q per unit length of the well. In deriving the equation, you will need to calculate the dispersion coefficient. You may take the molecular diffusion of radioactive material to be D.

(b) Explain how a localized release of radioactive material will spread over time.

(c) If now the rock absorbs some of the radiation at a rate λ times the concentration in the fluid, re-derive the governing equation for the conservation of the radiation, and calculate the steady profile of radiation in the water, and hence the profile of adsorbed radiation in the formation.

[*Hint: You may find it useful to know that*

$$\int_0^1 \left(\frac{z^3}{6} - \frac{z^4}{24} - \frac{z^2}{12}\right)\left(z - z^2 - 6\right)\, dz \;=\; 0.3525\,.]$$

4 (a) Briefly discuss the central assumptions of $k - \epsilon$ turbulence models.

(b) Consider a statistically stationary 1-D flow with no turbulent production, and a steady mean velocity $U_0 > 0$. The $k - \epsilon$ equations become

$$U_o \frac{dk}{dx} = \frac{d}{dx}\left(\frac{\nu_T}{\sigma_k}\frac{dk}{dx}\right) - \epsilon\,,$$
$$U_o \frac{d\epsilon}{dx} = \frac{d}{dx}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{d\epsilon}{dx}\right) - C_{\epsilon 2}\,\frac{\epsilon^2}{k}\,.$$

These equations have a front solution between non-turbulent flow ($k = \epsilon = 0$ for $x \leqslant 0$) and turbulent flow, which for small positive x has form

$$k = k_0\left(\frac{x}{\delta_0}\right)^p, \qquad \epsilon = \epsilon_0\left(\frac{x}{\delta_0}\right)^q, \qquad \delta_0 = \frac{k_0^{3/2}}{\epsilon_0}\,,$$

where p, q, k_0, ϵ_0 and δ_0 are positive constants. Remembering that $\nu_T = C_\mu k^2/\epsilon$ by assumption, show that

$$q = 2p - 1\,,$$
$$U_0 = \frac{C_\mu (2p-1)\,k_0^{1/2}}{\sigma_\epsilon}\,,$$
$$p = \frac{1}{2-\sigma}\,,$$

where $\sigma = \sigma_\epsilon/\sigma_k$.

(c) For small x, express k, ϵ, the integral length scale of the turbulence $L = k^{3/2}/\epsilon$, and the turbulent vorticity $\omega = \epsilon/k$ in terms of σ.

(d) Discuss what the dominant physical balance is for small x.

(e) Hence show that the model is self-consistent when $\sigma_k < \sigma_\epsilon < 1.5\sigma_k$.

(f) Finally, show that the gradient of the turbulent diffusivity at $x = 0_+$ can be expressed in terms of the steady mean velocity as

$$\left(\frac{d\nu_T}{dx}\right)_{x=0_+} = (2\,\sigma_k - \sigma_\epsilon)\,U_0\,.$$

5 (a) Ignoring boundary contributions, write down evolution equations for the total kinetic energy of a flow, the total potential energy of the flow, the background potential energy of a flow, and the available potential energy of a flow.

(b) Hence define the concepts of instantaneous and cumulative "mixing efficiency" in the context of stratified turbulent mixing.

(c) Consider a model for forced stratified turbulent mixing, where the buoyancy flux $\mathcal{B}$, the dissipation and the large-scale forcing are assumed to depend on the total kinetic energy K, and hence

$$\frac{d}{dt} K = F - \mathcal{B} - \epsilon = \frac{K^{1/2} U_0^2}{L_f} - \frac{K^{1/2} U_0^2}{L_\rho} - \frac{K^{3/2}}{L_v},$$

where L_f, L_ρ and L_v are appropriate characteristic integral length scales of the forcing, the density and velocity fluctuations respectively, and U_0 is an appropriate velocity scale of the forcing. Derive the time-dependent evolution of the kinetic energy.

(c) Assuming $L_f < L_\rho$, show that the kinetic energy approaches a steady state,

$$K_\infty = \frac{L_v}{L_f L_\rho} (L_\rho - L_f) U_0^2,$$

discussing the cases of "large" and "small" initial kinetic energies separately.

(d) Assuming that the exchange between the potential energy and the internal energy reservoirs is negligible, and also that the available potential energy is negligible, calculate the time-dependent instantaneous mixing efficiency and (in the limit of large time) the cumulative mixing efficiency.

(e) Discuss the two limiting cases $L_v \ll L_\rho$ and $L_v \gg L_\rho$, paying particular attention to physical interpretation.

6 (a) Define the concept of "pure plume balance" for plumes with non-zero source volume flux.

(b) Consider a room with cross-sectional area A_c and height H. Assume there is also a point-source turbulent plume with buoyancy flux πF_S at $z = 0$, and that the flow develops as a "filling box". Derive an expression for the time-dependent location of the "first front", i.e. the location of the lowest layer of fluid which has passed through the plume.

(c) Now assume that the plume has finite source volume flux $\pi Q_S > 0$ and specific momentum flux $\pi M_s > 0$, and an opening to the exterior at $z = h$ for $0 \leqslant h \leqslant H$. Derive differential equations for the evolution of the first front both above and below the height $z = h$, under the assumption that the plume is in pure plume balance at the source.

(d) Show that the first front reaches the floor $z = 0$ in finite time only if the opening to the exterior is at the floor $z = 0$, and calculate this arrival time.

(e) Finally, consider a room with openings with area A_H at $z = H$, and area A_0 at $z = 0$. Derive a condition for the entire room to be filled with fluid cycled through the plume in the limit of large time.

END OF PAPER