UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 7 June, 2004 1.30 to 4.30

PAPER 3

FINITE DIMENSIONAL LIE ALGEBRAS AND THEIR REPRESENTATIONS

Attempt **ALL** *questions.*

There are **five** *questions in total.*

Question 2 carries the most weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let $\mathfrak{g}$ be a Lie algebra. Define what it means for $\mathfrak{g}$ to be (a) solvable and (b) semisimple.

(ii) Let $\mathfrak{g} = \mathfrak{sl}_n$. Define two bilinear forms $(,)$ and $(,)_{ad}$ by $(A, B) = tr(AB)$ and $(A, B)_{ad} = tr(adA \cdot adB)$. Show $(,) = \lambda(,)_{ad}$ and compute λ.

(iii) Let $\mathfrak{g}$ be the four dimensional Lie algebra with basis c, d, p, q and Lie brackets

$$[c, \mathfrak{g}] = 0 \quad [p, q] = c \quad [d, p] = p \quad [d, q] = -q\,.$$

(a) Show $\mathfrak{g}$ is solvable.

(b) Construct a non-degenerate invariant bilinear form $(\, , \,)$ on $\mathfrak{g}$. Show there does not exist a representation V of $\mathfrak{g}$ such that $(x, y) = tr_V(xy)$.

(State clearly any theorems you use.)

2 Let $\mathfrak{g} = \mathfrak{so}_{2n} = \{A \in Mat_{2n} \mid AJ + JA^T = 0\}$, $\quad J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$, $\mathfrak{t}$ = diagonal matrices in $\mathfrak{g}$.

a) Decompose $\mathfrak{g}$ as a $\mathfrak{t}$-module, and hence write the roots R for $\mathfrak{g}$. Choose positive roots to be those ocurring in upper triangular matrices. Write the positive roots R^+, the simple roots Π, and the fundamental weights. Write ρ.

Draw the Dynkin diagram, and label it by the simple roots.

b) Show that $\mathfrak{so}_4$ is isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

c) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Describe the Weyl group W (you do not need to prove your answer).

d) Let $V = \mathbb{C}^{2n}$. Write the character of V, and determine its highest weight. Decompose $V \otimes V$ into irreducibles; determine their highest weights.

3 State the Weyl character formula, briefly defining the notation you use.

4 $B = \{\bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{1} \bullet\}$ is the crystal of a representation. Decompose $B \otimes B$ into irreducibles.

5 Let $\mathfrak{g} = \mathfrak{sl}_2$, and V be a respresentation of $\mathfrak{g}$.

(i) Show that if V is finite dimensional,

$$s = \exp(e)\exp(-f)\exp(e)$$

is well defined as a linear map $s : V \to V$. Here $\exp(x) = \sum_{n \geqslant 0} \frac{x^n}{n!}$ and e, f, h are the standard basis of $\mathfrak{s}l_2$.

(ii) Show that s is not well defined on a Verma module $M(n)$ with highest weight $n \in \mathbb{N}$.

(iii) Show that s^2 acts as $(-1)^n$ on $L(n)$, the irreducible representation with highest weight $n, n \in \mathbb{N}$.

[Hint: it is possible to do this without much computation.]