MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2008 1.30 to 4.30

PAPER 30

ERGODIC THEORY

Attempt no more than **THREE** *questions.*

There are **FOUR** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that X is a compact metric space and that $T : X \to X$ is a continuous map. What does it mean to say that the system (X, T) is minimal? Show that every system contains a nontrivial minimal subsystem, and deduce the Birkhoff Recurrence Theorem.

Suppose that (X, T) is a minimal system and that $(\tilde{X}, \tilde{T})$ is an extension of this system by the group $\mathbb{R}/\mathbb{Z}$ with cocycle $\rho : X \to \mathbb{R}/\mathbb{Z}$. State and prove a sufficient condition on ρ in order that $(\tilde{X}, \tilde{T})$ be a minimal system. Hence, or otherwise, show that the set of values taken by $n^2\sqrt{2}$, $n = 1, 2, 3, \ldots$, is dense in $\mathbb{R}/\mathbb{Z}$.

2 Suppose that (X, μ, T) is a measure-preserving system. What does it mean for the transformation $T : X \to X$ to be ergodic? State the pointwise ergodic theorem.

Let $X = [0, 1]$ and recall that the Gauss map $T : X \to X$ is defined by $Tx = \{1/x\}$ if $x \neq 0$ and $T0 = 0$. Define the Gauss measure ν, and show that T is measure-preserving and ergodic with respect to ν.

Show that for almost all real numbers $x \in [0, 1]$ the proportion of the partial quotients $a_1, a_2, \ldots, a_n$ in the continued fraction expansion for x which are equal to 2 tends to $\log_2 9 - 3$ as $n \to \infty$.

3 State and prove the Bochner-Herglotz spectral theorem [*you may assume the Fourier inversion formula for smooth functions*].

Using it, or otherwise, show that a measure-preserving system (X, μ, T) is weakly mixing if and only if it has no nonconstant eigenfunctions [*you may assume the L^2 ergodic theorem*].

Deduce that the $\times 2$ map on $\mathbb{R}/\mathbb{Z}$ is weakly mixing [*any results you use about the representation of functions $f \in L^2(\mathbb{R}/\mathbb{Z})$ as Fourier series may be used without detailed comment*].

4 What is meant by the assertion that a measure-preserving system has the SZ property at level k? Show that Szemerédi's theorem on arithmetic proressions is a consequence of the fact that every measure-preserving system has the SZ property at all levels [*any results you need on weak convergence of measures may be assumed without proof provided that they are properly stated*].

What does it mean to say that a measure-preserving system (X, μ, T) is compact? Prove that every compact system has the SZ property.

END OF PAPER