UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 1.30 to 4.30

PAPER 8

ELLIPTIC FUNCTIONS AND ELLIPTIC INTEGRALS

Attempt ***at most FOUR*** *questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) The lemniscate is the curve given by the equation $(x^2+y^2)^2 = x^2 - y^2$. Find a parametric form for the lemniscate, and show that its length L is given by

$$L = 4\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

(ii) Explain briefly why the function

$$F_1(z) = \int_0^z \frac{dw}{\sqrt{w(1-w^2)}}$$

is a conformal map of the upper half-plane $\mathbb{H}$ onto a square domain.

(iii) By considering the conformal map $g : z \mapsto (z-i)/(z+i)$ of $\mathbb{H}$ onto the unit disc $\mathbb{D}$, or otherwise, show that the lemniscatic integral

$$F(z) = \int_0^z \frac{dw}{\sqrt{1-w^4}}$$

is a conformal map of the disc $\mathbb{D}$ onto a square domain.

2 Let

$$p(z) = az^3 + bz^2 + cz + d, \quad q(z) = \alpha z^2 + \beta z + \gamma.$$

Define the resultant $R(p,q)$ of p and q. Explain the significance of the condition $R(p,q) \neq 0$.

Suppose that f is meromorphic in $\mathbb{C}$. Explain what is meant by the statement that f satisfies an algebraic addition theorem. Show that any polynomial satisfies an algebraic addition theorem. Briefly sketch the argument that any elliptic function satisfies an algebraic addition theorem.

By considering the function $z \mapsto \exp\big(\exp(z)\big)$, or otherwise, show that there is a simply periodic function that is analytic throughout $\mathbb{C}$ and that does not satisfy an algebraic addition theorem.

3 Write an essay on the construction of Jacobi's elliptic function sn that maps a rectangle onto the upper half-plane. You should include a brief discussion of the main points in the argument leading to the Schwarz-Christoffel formula, and show that, up to periodicity, for every w there are exactly two solutions of the equation $\mathrm{sn}(z) = w$.

Let $\wp$ be the Weierstrass elliptic function whose set of periods Λ coincides with that of sn. Show that there are complex numbers a, b, c, d and z_0, with $ad - bc \neq 0$, such that for all z,

$$\mathrm{sn}(z+z_0) = \frac{a\wp(z)+b}{c\wp(z)+d}.$$

4 The theta function θ_3 is defined by

$$\theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz},$$

where $q = \exp(\pi i \tau)$, and τ lies in the upper half-plane, and the function θ_4 is defined by $\theta_4(z) = \theta_3(z + \pi/2)$. Find the periods of the function θ_3/θ_4, and of the function $(\theta_3/\theta_4)^2$.

Find all zeros of the function θ_3.

5 Write an essay on the Modular group, one of its subgroups, and analytic functions invariant under one (or both) of these groups.