UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 1.30 to 4.30

PAPER 16

DYNAMICAL SYSTEMS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let A be an $m \times m$ matrix with zeros and ones and let Σ_A be the set of allowed two-sided sequences. Consider the subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$. Prove that:

(a) the number of fixed points of the shift σ in Σ_A is the trace of A;

(b) the number of allowed words of length $n+1$ beginning with the symbol i and ending with j is the i, j-th entry of A^n; and

(c) the number of periodic points of the shift σ of period n in Σ_A is the trace of A^n.

2 Let X be a compact metric space and $f : X \to X$ a continuous map.

(a) Define topological transitivity;

(b) Suppose that for any two non-empty open sets U and V, there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$. Show that f is topologically transitive.

(c) Show that if $f(X) = X$ and f is topologically transitive, then for any two non-empty open sets U and V, there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$. Is the result still true if we drop the condition $f(X) = X$?

3 Let X be a compact metric space and $f : X \to X$ a continuous map.

(a) Show that the topological entropy of f does not depend on the particular choice of metric generating the topology of X.

(b) Show that topological entropy is an invariant of topological conjugacy.

(c) Let A be a hyperbolic toral automorphism of the 2-torus. Find the topological entopy of A. Justify your answer.

4 (a) State the Birkhoff ergodic theorem.

(b) Given an integer m with $m \geqslant 2$, consider the expanding map $E_m : S^1 \to S^1$ given by

$$E_m(x) = mx \;\; mod\, 1.$$

Show that E_m is ergodic with respect to Lebesgue measure.

(c) Show that for almost every $x \in [0, 1)$ (with respect to Lebesgue measure) the frequency of 1's in the binary expansion of x is 1/2.

5 Let X be a compact metric space and $f : X \to X$ a continuous map. A sequence of points $x_0, x_1, x_2, \ldots$ in X is said to be *evenly distributed* with respect to a Borel probability measure μ, if the following condition is satisfied: for every continuous function $\varphi : X \to \mathbf{R}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i)$$

must exist and be equal to the space average

$$\int_X \varphi \, d\mu.$$

Show that if the forward orbit of x is evenly distributed for μ-almost every x, then μ is f-invariant and ergodic. [*You may use that μ is invariant if and only if for every continuous φ,*

$$\int_X \varphi \circ f \, d\mu = \int_X \varphi \, d\mu.$$

To prove ergodicity you may assume:

(i) Birkhoff ergodic theorem;

(ii) given any Borel set S, $\mu(S)$ is the supremum of $\mu(K)$ as K varies over compact sets of S. Also, $\mu(S)$ is the infimum of $\mu(U)$ as U varies over open sets containing S;

(iii) given an open set U and a compact set $K \subset U$, there exists a continuous function $\varphi : X \to [0,1]$ which takes the value 1 on K and the value 0 outside U.]

END OF PAPER