

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 11

PAPER 29

DIOPHANTINE ANALYSIS AND TRANSCENDENCE THEORY

*Attempt **TWO** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

*Cover sheet
Treasury Tag
Script paper*

SPECIAL REQUIREMENTS

None

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 Prove that π is transcendental. State the more general Lindemann theorem and show that it implies that $\cos \alpha \cos \beta$ is transcendental for all non-zero algebraic α, β .

2 State and prove Siegel's lemma on the solution of linear equations.

Define Siegel's E -functions. State the Siegel-Shidlovsky theorem and specify the particular E -functions that give Lindemann's theorem.

3 Using Mahler's method, show that the Fredholm series $\sum_{n=0}^{\infty} z^{l^n}$, where l is an integer ≥ 2 , assumes transcendental values for all algebraic z with $0 < |z| < 1$.

Prove that, for $l \geq 3$ and $z = \frac{1}{2}$, the transcendence of the Fredholm series can be obtained directly from the Thue-Siegel-Roth theorem.

4 State the Gelfond-Schneider theorem and outline a proof.

Show that the theorem implies that if α, β are algebraic numbers, not 0 or 1, then $\log \alpha / \log \beta$ is either rational or transcendental. State a generalisation applicable to arbitrarily many logarithms.

5 Assuming an appropriate estimate for a logarithmic form, show that the equation $\alpha x + \beta y = 1$, where α, β are non-zero elements of an algebraic number field K , has only finitely many solutions in units x, y in K . Indicate how this leads to a proof of Thue's theorem on the equation $F(x, y) = m$.

6 State the abc -conjecture. Show that it implies that the Fermat equation

$$x^n + y^n = z^n$$

has only finitely many solutions in positive integers x, y, z and $n > 3$.

Show further that the conjecture implies that the exponential equation

$$ax^n - by^n = c,$$

where a, b, c are given positive integers, has only finitely many solutions in positive integers x, y and $n > 2$. Indicate how the latter can be established from the theory of linear forms in logarithms without any unproved hypothesis.

END OF PAPER