MATHEMATICAL TRIPOS Part III

Monday 2 June 2008 1.30 to 4.30

PAPER 19

DIFFERENTIAL GEOMETRY

*Attempt **THREE** questions*

*There are **FIVE** questions in total*

The questions are of equal weight

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what is meant by a vector field on a smooth manifold M. Given vector fields X, Y on M, define the *Lie bracket* $[X, Y]$ as a vector field on M. If $f : M \to N$ is a diffeomorphism of smooth manifolds, define the operator f_* on vector fields, and prove that $f_*[X, Y] = [f_*X, f_*Y]$.

Suppose that G is a Lie group; what does it mean to say that a vector field X on G is left-invariant? Prove the existence of a unique left-invariant vector field with a given value $\xi \in T_eG$ at the identity, and deduce the existence of a Lie-bracket on T_eG induced from the Lie bracket on vector fields.

Suppose now that $G = GL(n, \mathbf{R}) \subset M_{n\times n}(\mathbf{R})$. Choosing suitable coordinates x_{pq} for $M_{n\times n}(\mathbf{R})$, and hence an identification of each tangent space with $M_{n\times n}(\mathbf{R})$, show that the induced Lie bracket on T_eG corresponds to the commutator of matrices.

2 Given a vector bundle $\pi : E \to M$ over a smooth manifold M of dimension n, and r a positive integer, describe briefly the construction of the rth exterior power bundle $\Lambda^r E$ over M. Assuming the existence of partitions of unity, show that the line bundle $\Lambda^n T^*M$ is trivial if and only if there exists a family of charts $\{U_\alpha\}$ in the given differential structure on M for which the corresponding coordinate domains cover M and the Jacobian matrices on the overlaps all have positive determinant. (If these conditions are satisfied, then the manifold is called *orientable.*)

If $\pi : E \to M$ is a vector bundle of rank r over an orientable manifold M, prove that the total space of the bundle is an orientable manifold if and only if the line bundle $\Lambda^r E$ over M is trivial. Given any smooth manifold M, show that the total space of the cotangent bundle T^*M is always an orientable manifold.

3 Define the exterior derivative map d on r-forms on a smooth manifold M $(r \geq 0)$. If ω is a 1-form and X, Y are vector fields with Lie bracket $[X, Y]$, show that

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Define what is meant by a *connection* D on a vector bundle E over M, and by the corresponding *covariant exterior derivative* d^E. Define also the *curvature* R, explaining why it may be considered as a section of $\Lambda^2 T^*M \otimes \mathrm{End}(E)$. For any section σ of E, show that

$$R(X, Y)(\sigma) = D_X D_Y \sigma - D_Y D_X \sigma - D_{[X,Y]}\sigma.$$

Given a connection D on E, show that the recipe $(\tilde{D}_X \alpha)(s) = D_X(\alpha(s)) - \alpha(D_X s)$ for all sections α of $\mathrm{End}(E)$, sections s of E, and vector fields X, defines an induced connection $\tilde{D}$ on the bundle $\mathrm{End}(E)$. Prove, for all vector fields X and Y, the identity

$$\big([\tilde{D}_X, \tilde{D}_Y]\alpha\big)(s) = [D_X, D_Y](\alpha(s)) - \alpha([D_X, D_Y]s),$$

where the square brackets here denote commutators of vector bundle endomorphisms. Deduce that the curvature $\tilde{R}$ of $\tilde{D}$ is zero if and only if $R = \omega \otimes \mathrm{id}$ for some 2-form ω, where id denotes the identity endomorphism of E.

4 Given a smooth curve $\gamma : [a, b] \to M$ on a smooth manifold, define what is meant by a *smooth vector field* $V(t)$ *along* γ. Given a Koszul connection ∇ on M, explain carefully the concepts of the *covariant derivative* of $V(t)$ along γ, and $V(t)$ being *parallel* along γ. Given a tangent vector $V_a \in T_{\gamma(a)}M$, show that there exists a unique parallel vector field $V(t)$ along γ with $V(a) = V_a$. Hence deduce the existence of the parallel translation maps $\tau_t : T_{\gamma(a)}M \to T_{\gamma(t)}M$, which are isomorphisms of vector spaces.

Given a Riemannian metric g on M, what does it mean to say that a Koszul connection ∇ is a *metric connection*? Deduce in this case that the parallel translation maps are isometries with respect to the given inner-products on tangent spaces.

Given a piecewise smooth closed curve $\gamma : [a, b] \to M$, and ∇ a metric connection, show that parallel translation determines an orthogonal map τ on the space $T_{\gamma(a)}M$. A smooth curve γ is called a *geodesic* if the smooth vector field $\dot{\gamma}(t)$ along γ is parallel. Find an example of a closed geodesic on a surface giving rise to an orthogonal map τ which is a reflection. Assuming that the great circles on S^2 parametrized with constant speed $\|\dot{\gamma}\|$ with respect to the standard Riemannian metric are geodesics with respect to the Levi-Civita connection, consider the case when γ is the concatenation of the sides (parametrized with unit speed) of a spherical triangle: determine the orthogonal map τ in terms of invariants of the triangle.

5 Let ∇ denote the Levi-Civita connection on a Riemannian manifold M and R denote the Riemannian curvature tensor. For X, Y, Z, W vector fields on M, write down (without proof) the full list of symmetries satisfied by $R(X, Y, Z, W)$. Define what is meant by the *Ricci tensor*, proving that it is a rank 2 symmetric tensor on M.

Define the *sectional curvatures* and the *Ricci curvatures*. When $\dim M = 3$, show that, if the Ricci curvatures are constant at some point, then so too are the sectional curvatures.

If (M, g_M) and (N, g_N) denote Riemannian manifolds with metrics g_M and g_N respectively, show that there is a corresponding metric $g_M \oplus g_N$ on $M \times N$. Identifying vector fields on M or N as vector fields on the product $M \times N$, show that, if X is a vector field on M and Y a vector field on N, then

(i) the Lie bracket $[X, Y] = 0$ on $M \times N$,
(ii) $\nabla_X Y = 0$, where ∇ denotes the Levi-Civita connection on $(M \times N, g_M \oplus g_N)$, and
(iii) $R(X, Y, X, Y) = 0$, for R the corresponding Riemannian curvature tensor.

Show that there is a Riemannian metric on $S^2 \times S^2$ for which the Ricci curvatures are constant, but the sectional curvatures are not.

[You may assume that the Christoffel symbols of the Levi-Civita connection on a Riemannian manifold (M, g) are given, with respect to a local coordinate system $x_1, \ldots, x_m$, by

$$\Gamma_{ij}^l = \frac{1}{2} \sum_k g^{lk} \big(\partial g_{jk} / \partial x_i + \partial g_{ki} / \partial x_j - \partial g_{ij} / \partial x_k \big).$$

Any facts you may need concerning the curvature of S^2 may also be assumed.]

END OF PAPER