UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2002 9 to 12

PAPER 14

DIFFERENTIAL GEOMETRY

*Attempt **THREE** questions, at most **ONE** of which should be from Section A*

*There are **two** questions in Section A and **four** in Section B*

The questions carry equal weight

All the manifolds and related concepts should be assumed to be smooth.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Define the notion of a Lie group. Show that the special unitary group $SU(n)$ is a real manifold, by constructing an appropriate family of charts. Hence show that $SU(n)$ is a Lie group and find its dimension. Prove rigorously that $SU(2)$ is diffeomorphic to S^3.

[*Standard results on the exponent and logarithm of matrices may be used without proof provided these are accurately stated.*]

2 Give three different formulations of 'orientability' for a connected manifold. Prove that the three formulations are equivalent and explain what is meant by an 'orientation' in each case.

Prove that the sphere S^n is orientable for every n. By considering a $2:1$ map $S^n \to \mathbb{R}P^n$, or otherwise, determine all the dimensions n for which the projective space $\mathbb{R}P^n$ is orientable.

[*You may use without proof any preliminary results you require provided that you state them accurately.*]

SECTION B

3 Give a definition of an embedded submanifold. State the Inverse Mapping Theorem for maps between open domains in $\mathbb{R}^n$. Define what is meant by a regular value of a smooth map $f : M \to N$ between manifolds M and N and prove that the inverse image of a regular value of f is an embedded submanifold of M of dimension $\dim M - \dim N$.

Show that $X = \{\, x_0{:}x_1{:}x_2 \in \mathbb{R}P^2 \mid x_2 = 0\}$ is an embedded submanifold of $\mathbb{R}P^2$ diffeomorphic to $\mathbb{R}P^1$, but there is no smooth function $f : \mathbb{R}P^2 \to \mathbb{R}$, with 0 a regular value of f and such that $X = f^{-1}(0)$.

4 Define what is meant by a complex vector bundle E of rank k over a manifold B. Give a definition of the transition functions of E, including the cocycle conditions satisfied by these functions and their relation to a structure group G of E. Explain how a principal G-bundle over B may be constructed from the transition functions of E.

Let E be a rank 1 complex vector bundle over $\mathbb{C}P^1$, such that the complement of the zero section of E is diffeomorphic to $\mathbb{C}^2 \setminus \{(0,0)\}$, with the projection given by $(z_1, z_2) \to z_1 : z_2$. (You may assume without proof that such E exists.) Find a set of transition functions making E into a vector bundle with structure group $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. Deduce that the principal S^1-bundle associated to E may be given as $S^3 \to S^2$.

5 Define a Riemannian metric and the Levi–Civita connection on a Riemannian manifold. Prove that there exists unique Levi–Civita connection on any Riemannian manifold.

Define the curvature $(R_{ij,kl})$ of a Riemannian metric g and give expressions for the Ricci curvature (Ric_{ij}) and for the scalar curvature s in terms of the curvature and the metric. Show that g has constant scalar curvature if and only if $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$.

6 Define the volume form on an oriented Riemannian manifold. Explain why the definition is independent of the choice of local coordinates. Define the Hodge star operator on differential forms and compute its square.

Now let M be a closed (compact and without boundary) oriented n-dimensional Riemannian manifold. The linear operator $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ is defined by $\delta = (-1)^{n(p+1)+1} * d *$ for $p > 0$ and $\delta = 0$ for $p = 0$. Using the fact that $\int_M d\psi = 0$ for every differential $(n-1)$-form ψ on M, deduce that $\int_M (\beta \wedge *d\alpha) = \int_M ((\delta\beta) \wedge *\alpha)$, for every differential p-form α and $(p+1)$-form β on M.

Define the Laplace–Beltrami operator Δ and harmonic differential forms on M. Show that for every harmonic differential form α, $d\alpha = \delta\alpha = 0$. State the Hodge Decomposition Theorem and deduce from it that any exact harmonic form is zero.