MATHEMATICAL TRIPOS **Part III**

Tuesday 10 June 2008 1.30 to 4.30

PAPER 2

DECISION PROBLEMS IN GROUP THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let $G = \langle g : g^4 = 1 \rangle$ and $H = \langle h : h^{10} = 1 \rangle$.

Let $G_0 = \langle g^2 \rangle \leqslant G,\ H_0 = \langle h^5 \rangle \leqslant H$ and $\phi : G_0 \cong H_0$.

Prove that $A := G * H\ (\phi : G_0 \cong H_0)$ has soluble word problem clearly stating any facts about free products with amalgamated subgroups that you use.

Solve the word problem for $A/\langle [g,h] \rangle^A$.

(ii) Let $N = \langle a, b : [a,b,a] = 1 = [a,b,b] \rangle$. Give an algorithm to show that every element of N can be reduced to an element of the form $a^m b^n\, [a,b]^k$ with $m, n,\, k \in \mathbb{Z}$.

[*You may assume that* $[b^n, a^m]^{-1} = [a^m, b^n] = [a,b]^{mn}$ *and* $b^n a^m = a^m b^n\, [b^n, a^m]$ *for all* $m, n \in \mathbb{Z}$.]

Assuming that the form $a^m b^n [a,b]^k$ is unique, solve the word problem for N. What is the nilpotency class of N?

2 (i) Define what is meant by a primitive recursive function and prove from your definition that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(m,n) = m^n + n$ is primitive recursive.

(ii) Define what is meant by a recursive function and by a recursively enumerable set.

(iii) State and prove the undecidability of the Halting Problem. Deduce that there is a recursively enumerable set that is not recursive.

[*You may assume Church's Thesis whenever you need it.*]

3 (i) Prove that every finitely presented group can be embedded in a 2 generator finitely presented group clearly stating any results that you assume.

(ii) State the Messuage Lemma for finitely presented groups. Use it to determine if there is an algorithm to determine whether or not an arbitrary finitely presented group is infinite, proving any intermediate result you may need.

[*You may assume that there is a finitely presented group with insoluble word problem.*]

4 State the Higman Embedding Theorem. Use it to prove:

(i) There is a finitely presented group G_0 with insoluble word problem.

(ii) Let G_0 in (i) be given by $F/\langle\{r_j : j \in J\}\rangle^F$, where F is a free group on a finite set X and J is finite. Let $H = \langle\{(x,x) : x \in X\} \cup \{(1,r_j) : j \in J\}\rangle \leqslant F \times F$. Prove that the membership problem for H (in $F \times F$) is undecidable.

5 Give an algebraic characterisation for a finitely generated group to have soluble word problem.

Prove your characterisation is true clearly stating any theorems that you use.

6 Let G be a finitely generated group and H a subgroup of G. Define what is meant by: H is a benign subgroup of G. If H, K are benign subgroups of G and G can be embedded in a finitely presented group, prove that $H \cap K$ and $\langle H, K\rangle$ are benign subgroups of G. If $N \lhd G$ and is benign, prove that G/N can be embedded in a finitely presented group.

END OF PAPER