UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2006 1.30 to 4.30

PAPER 26

CYCLOTOMIC FIELDS

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Notation: Throughout, p will denote an odd prime number, $\mathbb{Z}_p$ the ring of p-adic integers, and $R = \mathbb{Z}_p[[T]]$ the ring of formal power series in an indeterminate T with coefficients in $\mathbb{Z}_p$. For each $n \geqslant 0$, ζ_n will denote a primitive p^{n+1}-th root of unity with $\zeta_{n+1}^p = \zeta_n$. We let K_n be the field $\mathbb{Q}(\zeta_n)$. For each prime q, μ_q will denote the group of q-th roots of unity.

1 Prove that there exists a unique map $\psi : R \to R$ such that, for all $f \in R$,

$$\psi(f)\left((1+T)^p - 1\right) = \frac{1}{p}\sum_{\xi \in \mu_p} f\left(\xi(1+T) - 1\right).$$

Prove that, for all integers $n \geqslant 1$, we have

$$\psi\left(T^{np} + T^{np-1}\right) \equiv T^n + T^{n-1} \, mod \, p \, R.$$

2 Let a, b be integers with $(a, p) = (b, p) = 1$, and, for each integer $n \geqslant 0$, define

$$c_n(a,b) = \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}}.$$

Prove that $N_{n,n-1}(c_n(a,b)) = c_{n-1}(a,b)$ for all $n \geqslant 1$, where $N_{n,n-1}$ denotes the norm map from K_n to K_{n-1}. Put $c(a,b) = (c_n(a,b))$, and compute the higher logarithmic derivatives

$$\delta_k\left(c(a,b)\right)$$

for all $k \geqslant 1$.

3 Define an Euler system for the tower of fields $K_n^+ (n = 0, 1, 2, \ldots)$, where K_n^+ denotes the maximal real subfield of K_n. Prove that every cyclotomic unit in this tower gives rise to an Euler system, verifying in detail the three axioms for the Euler system in this case.

4 Write an essay sketching the proof of the main conjecture on cyclotomic fields via Iwasawa's theorem.

END OF PAPER