UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2008 1.30 to 4.30

PAPER 62

COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The universe today appears to be well-approximated by a flat FRW universe containing radiation, matter, and a cosmological constant.

(a) Show that Friedmann's equation may be written

$$a'^2 = H_0^2 \left(\Omega_r + \Omega_m a + \Omega_\Lambda a^4\right), \qquad (*)$$

where $a(t)$ is the scale factor normalised to unity today, and prime denotes derivative with respect to conformal time $\tau = \int_0^t dt/a(t)$. Explain the physical significance of the parameters H_0, Ω_r, Ω_m, and Ω_Λ. What are the approximate observed values for these parameters today? [From CMB observations, we have $\Omega_r = 0.42 \times 10^{-4}\, h^{-2}$]

(b) Consider the regime of small $a \ll (\Omega_m/\Omega_\Lambda)^{\frac{1}{3}}$, in which the Ω_Λ term in (*) can be neglected. Show, by using (*) to represent τ as an integral over a, or otherwise, that

$$\tau \approx 2H_0^{-1}\Omega_m^{-1}\left(\sqrt{\Omega_r + \Omega_m a} - \sqrt{\Omega_r}\right), \qquad a \ll (\Omega_m/\Omega_\Lambda)^{\frac{1}{3}}.$$

Show that in the same approximation, the conformal time at equal matter and radiation density

$$\tau_{eq} = 2H_0^{-1}\Omega_m^{-1}\Omega_r^{\frac{1}{2}}(\sqrt{2} - 1).$$

Explain why this represents the comoving horizon scale at that time and estimate this comoving scale in Megaparsecs. [You may use $H_0^{-1} \approx 3000h^{-1}$ Mpc.]

(c) For $a \gg \Omega_r/\Omega_m$, the Ω_r term is negligible in (*). Use this to show that the conformal time today,

$$\tau_0 \approx H_0^{-1}\Omega_m^{-\frac{1}{3}}\Omega_\Lambda^{-\frac{1}{6}} \int_0^{(\Omega_\Lambda/\Omega_m)^{1/3}} \frac{dy}{\sqrt{y(1+y^3)}}.$$

(d) If $\Omega_\Lambda/\Omega_m \gg 1$, the integral can be approximated by

$$I = \int_0^\infty \frac{dy}{\sqrt{y(1+y^3)}}\,.$$

Show that in this approximation, and assuming $\tau_{eq} \ll \tau_0$, the angle subtended today by the comoving horizon scale at equal matter and radiation density,

$$\theta \approx 2I^{-1}(\sqrt{2} - 1)\Omega_r^{\frac{1}{2}}\Omega_\Lambda^{\frac{1}{6}}\Omega_m^{-\frac{2}{3}}.$$

2 The general relativistic action for a scalar field ϕ is given by

$$\mathcal{S}_M = \int d^4x\sqrt{-g}\big(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\big).$$

(a) Show that the stress-energy tensor, $T^{\mu\nu} = (2/\sqrt{-g})(\delta\mathcal{S}_M/\delta g_{\mu\nu})$, is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\big).$$

(b) Consider a flat FRW universe, $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$. Show that the energy density $T^{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}a^{-2}(\nabla\phi)^2 + V(\phi)$, the momentum density $T^{0i} = -a^{-2}\dot{\phi}\partial_i\phi$ and the spatial stress $T^{ij} = a^{-4}\partial_i\phi\partial_j\phi + \delta_{ij}a^{-2}\big(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}a^{-2}(\nabla\phi)^2 - V(\phi)\big)$.

(c) Show that the equation of motion of the scalar field is

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - a^{-2}\vec{\nabla}^2\phi = -\frac{\partial V}{\partial\phi}\,.$$

(d) If the scalar field is spatially homogeneous, explain why the stress-energy tensor becomes spatially isotropic and so we can write $T^{ij} = Pa^{-2}\delta_{ij}$. Show how covariant stress-energy conservation then reduces to $\dot{\rho} = -3\frac{\dot{a}}{a}(P+\rho)$, where $\rho = T^{00}$. [You may use $\Gamma^0_{ij} = \delta_{ij}a\dot{a}$ and $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_{ij}$, with all other components of the Christoffel symbol being zero.]

(e) Show that a spatially homogeneous free scalar field, with $V(\phi) = 0$, behaves like a perfect fluid with $P = \rho$. Show that $a(t) \propto t^{1/3}$ for a universe dominated by such a field.

3 For cosmological nucleosynthesis,

(i) Explain why the rate of the weak interactions which convert protons to neutrons and vice versa is given, up to a numerical factor, by $G_F^2 T^5$. [The Fermi constant $G_F = (\sqrt{2}g^2)/(8M_W^2) \approx 1.17 \times 10^{-5} GeV^{-2}$, where g is the weak $SU(2)$ coupling and M_W is the mass of the W-boson.]

(ii) Show that the weak interactions freeze out at a temperature approximately given by

$$T_F \sim G_F^{-2/3} \mathcal{N}_{eff}^{\frac{1}{6}} M_{Pl}^{-1/3},$$

where $M_{Pl} = 1/\sqrt{8\pi G}$ and $\mathcal{N}_{eff}$ is the effective number of relativistic degrees of freedom, $\mathcal{N}_{eff} = \mathcal{N}_{bose} + \frac{7}{8}\mathcal{N}_{fermi}$.

(iii) What is $\mathcal{N}_{eff}$ at these epochs, when photons as well as electrons, neutrinos and their antiparticles are relevant? (You should assume three relativistic neutrino species.) Estimate T_F, assuming it to be greater than the electron mass. How does $\mathcal{N}_{eff}$ change as the universe cools through $e^{\pm}$ annihilation, and what is the relation between the photon and neutrino temperatures after this event?

(iv) How does the neutron/proton ratio at weak interaction freeze-out depend on T_F? [The neutron-proton mass difference is $Q = 1.29$ MeV, and you may assume the chemical potentials μ_n and μ_p are negligible.]

(v) In thermal equilibrium the mass fraction of a nuclear species of mass number A and proton number Z is proportional to

$$\eta^{A-1} X_p^Z X_n^{A-Z} e^{B_A/T},$$

where X_p and X_n are the proton and neutron mass fractions, $\eta = n_B/n_\gamma$ is the baryon-to-photon ratio and B_A is the binding energy of the species concerned.

Explain the effect on the final ^4He mass fraction of

(a) increasing the number of neutrino species,

(b) increasing the neutron half-life $\tau_{1/2}(n)$,

(c) increasing η.

4 Consider a flat FRW universe containing cold dark matter (CDM) and radiation. The fractional density perturbations in the CDM and the radiation are expanded in Fourier modes: $\delta_C(\mathbf{x}) = \sum_{\mathbf{k}} \delta_{C,\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}$, and similarly for $\delta_R(\mathbf{x})$, with each Fourier mode obeying

$$\delta''_{C,\mathbf{k}} + \frac{a'}{a}\delta'_{C,\mathbf{k}} = 4\pi G a^2(\rho_C \delta_{C,\mathbf{k}} + 2\rho_R \delta_{R,\mathbf{k}}), \qquad (1)$$

where primes denote conformal time derivatives, ρ_C and ρ_R are the background CDM and radiation densities.

For adiabatic perturbations, while a perturbation mode with wavenumber k is still outside the horizon, $k\tau \ll 1$, the relation $\delta_{R,\mathbf{k}} \approx \frac{4}{3}\delta_{C,\mathbf{k}}$ holds.

(a) Show that in both the radiation-dominated era, with $a \propto \tau$, and the matter-dominated era, with $a \propto \tau^2$, there is a super-horizon growing mode solution $\delta_{C,\mathbf{k}} \propto \tau^2$.

(b) When $k\tau$ grows larger than unity, the mode crosses the horizon. In the radiation-dominated era, $\delta_{R,\mathbf{k}}$ then starts to oscillate about zero and its contribution to the right hand side of (1) may thereafter be neglected. Show that for such a mode, after horizon crossing but still in the radiation era, $\delta_{C,\mathbf{k}}$ has a growing solution proportional to $\ln(\tau)$.

(c) Once the universe enters the matter-dominated era, the ρ_R term in (1) may be neglected. Show that in this era, $\delta_{C,\mathbf{k}}$ grows as τ^2 for all k.

(d) The Harrison-Peebles-Z'eldovich (HPZ) spectrum is defined by the initial conditions

$$\langle|\delta_{C,\mathbf{k}}|^2\rangle \sim \mathcal{C}\tau^4 k, \qquad k\tau < 1,$$

early in the radiation epoch. The fractional mass perturbation $(\delta M/M)_k$ in a sphere of radius $\sim k^{-1}$ has a variance given approximately by $\langle(\delta M/M)_k^2\rangle \sim k^3\langle|\delta_{C,\mathbf{k}}|^2\rangle$. Show that for the Harrison-Peebles-Z'eldovich spectrum, $\langle(\delta M/M)_k^2\rangle$ is approximately constant at horizon crossing.

(e) Show that in the matter era, the HPZ spectrum retains the same shape for $k\tau_{eq} < 1$, but for larger k bends over to

$$\langle|\delta_{C,\mathbf{k}}|^2\rangle \sim \mathcal{C}(\frac{\tau}{\tau_{eq}})^4 k^{-3}\left(\ln(k\tau_{eq})\right)^2, \qquad k\tau_{eq} \gg 1,$$

where τ_{eq} is the conformal time at equal matter and radiation densities. Explain this bend in the spectrum around $k \sim \tau_{eq}^{-1}$ in terms of the growth behaviour computed above, and estimate $\langle(\delta M/M)_k^2\rangle$ for $k\tau_{eq} > 1$. Hence estimate the magnitude of the constant $\mathcal{C}$ which is required in order that perturbations on comoving scales of a few Megaparsecs reach $(\delta M/M)_k^2$ of order unity by today.

END OF PAPER