UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 9 to 11

## PAPER 56

## COSMOLOGY

*Attempt* **THREE** *questions.*

*There are* **four** *questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** In the early universe, the Friedmann equation is

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}\,.$$

Briefly discuss the evolution of the universe for different values of $k$.

The Raychaudhuri equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}\,.$$

In a FRW spacetime the decelaration parameter is defined by

$$q = -\frac{\ddot{a}a}{\dot{a}^2}\,.$$

In a universe with a cosmological constant show that

$$q = \frac{1}{2}\Omega_M - \Omega_\Lambda$$

in the matter era and

$$q = \Omega_R - \Omega_\Lambda$$

in the radiation era where $\Omega_M, \Omega_R$ and $\Omega_\Lambda$ are the density parameters. Discuss the significance of $q$.

In conformal time the Hubble parameter is $\mathcal{H}$. Show that

$$\Omega_M' = \mathcal{H}\Omega_M(\Omega_M - 1)$$

in the matter era, where prime denotes derivatives with respect to conformal time. Briefly discuss the flatness problem.

**2** In the non-relativistic limit the number density for a species of mass $m$ and chemical potential $\mu$ is

$$n = g\Big(\frac{mT}{2\pi}\Big)^{3/2} \quad e^{-(m-\mu)/T}\,,$$

where $g$ is the number of degrees of freedom and $T$ is the temperature. By considering the number densities of electrons, protons and hydrogen atoms prior to photon decoupling derive Saha's equation

$$\frac{1-X_e}{X_e^2} = \frac{2\xi(3)}{\pi^2}\eta\Big(\frac{2\pi T}{m_e}\Big)^{3/2} \; e^{I/T}$$

where $X_e = n_e/n_B$ is the fractional ionisation of electrons, $n_B$ is the number density of baryons, $\eta = n_B/s$ and $I = m_p + m_e - m_H$, with $m_p, m_e, m_H$ being the respective masses for protons, electrons and hydrogen.

Qualitatively discuss recombination, photon decoupling and residual ionisation. Why is the recombination temperature lower than the reionisation temperature of hydrogen?

**3** In the early universe the equations of motion of a scalar field are

$$H^2 = \frac{8\pi G}{3}\Big[\frac{1}{2}\dot\phi^2 + V(\phi)\Big]$$
$$\ddot\phi + 3H\dot\phi + \frac{dV}{d\phi} = 0$$

where $V(\phi)$ is the potential. Discuss the conditions necessary for 'vacuum domination' and 'slow roll'. For the potential $V(\phi) = \frac{1}{2}m^2\phi^2$ use the slow roll approximation to obtain the inflationary solutions,

$$\phi(t) = \phi_i - \frac{m m_{pl}}{2\sqrt{3\pi}}t$$
$$a(t) = a_0 \exp\frac{2\pi}{m_{pl}^2}[\phi_i^2 - \phi^2(t)]$$

where $\phi_i$ is the value of the field at the start of inflation and $m_{pl}$ is the Planck mass. What is the number of e-folds?

What is the value of $\phi$ when inflation ends? If $V(\phi_i) \sim m_{pl}^4$ estimate the overall expansion due to inflation.

**4** A theory of the weak interactions predicts the existence of an extra species of massless neutrino with interaction rate $\Gamma = G_A^2 T^5$, where $G_A \sim 10^{-12} GeV^{-2}$. Estimate the approximate temperature $T_D$ at which these neutrinos decouple from thermal equilibrium. What is the temperature today of these neutrinos relative to the photon background? Estimate the proportion of the total entropy of the universe today in these new neutrinos/anti-neutrinos.

If these new neutrinos were found to have a small mass, discuss the constraints on the mass imposed by standard cosmology.

[*You may assume that the effective massless degrees freedom at high energy is* $g^* = 100$ *and just before decoupling of ordinary neutrinos* $g^* = 10.75$.]